Quaternions in Computer Vision and Robotics

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Abstract

Computer vision and robotics suffer from not having good tools for manipulating three-dimensional objects. Vectors, coordinate geometry, and trigonometry all have deficiencies. Quaternions can be used to solve many of these problems. Many properties of quaternions that are relevant to computer visions and robotics are developed. Examples are given showing how quaternions can be used to simplify derivations in computer vision and robotics.

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1. Introduction

In computer vision and robotics, the nature of the mathematical tools available makes a large difference in the kind of things that can be done, both in theory and in practice. In deriving any relationship in computer vision, the researcher is often daunted if a large system of equations develops, and sometimes gives up. Formulation of equations is important in practice also: for example, in simulating the motion of a robot arm for the purpose of prediction, the complexity of the equations has a large influence on how fast the simulation can be done. So any tool which reduces the complexity of equations in a derivation or simulation must be seen as useful.

Several different systems have been used to describe positions and motions in space in computer vision and robotics: they are three-dimensional vectors, three-dimenstional coordinates, and trigonometry. Each of these has particular advantages and disadvantages. Vectors are the most elegant system, but unfortunately they are incomplete: certain operations, e.g., rotation, are not easily represented using vectors. Three-dimensional coordinates are complete, but often lead to lengthy and messy derivations, with many repetitive terms. Trigonometry is often quite useful in illuminating an otherwise difficult to see relationship (for example, Kanade's derivation of the "skewed symmetry constraint" [2]) but here the derivations can be even messier, requiring clever use of half-angle relationships.

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What is needed is a tool which is as powerful as vector notation, but which allows the representation of operations not directly representable with vectors, such as rotations. The mathematical object called "quaternion" is such a tool.

Quaternions were invented by Hamilton in the early 1840's [1]. They were the result of an attempt by Hamilton to resolve the question: What is the result of dividing one (three-dimensional) vector by another? The story [3] goes that Hamilton thought about this question for some time, then while walking across a bridge he saw the answer, and carved in the stone the formula that was the basis for quaternions:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1 \tag{1}$$

This formula gives the rule for multiplying two quaternions. What Hamilton had discovered is that while it is not possible to create a three-dimensional system (i.e., one consisting only of three-vectors) that enjoys a reasonable number of properties of the real and complex numbers, in four dimensions this is possible: in quaternions, all properties of the real and complex numbers are preserved except for commutativity of multiplication. Moreover, quaternions can be used to represent many operations in three-dimensional space, including rotations, affine transformations, and projections.

There are several equivalent ways of writing quaternions in terms of their four components; one way that is particularly useful is what Hamilton called Standard Quadrinomial Form:

$$\mathcal{Q} = \{\alpha + \beta\mathbf{i} + \gamma\mathbf{j} + \delta\mathbf{k} : \alpha, \beta, \gamma, \delta \, \text{real}\}.$$

In this system, Equation 1 gives the rule for multiplications, so that $\mathbf{ij} = \mathbf{k}$ but $\mathbf{ji} = -\mathbf{k}$. (Obviously multiplication is not commutative here.) These properties of complex and real numbers hold for the set of all quaternions $\mathcal{Q}$ as well:

1. Addition:
a. Closure: if $P, Q \in \mathcal{Q}$ then $P + Q \in \mathcal{Q}$
b. Commutativity: $P + Q = Q + P$ for all $P, Q \in \mathcal{Q}$
c. Associativity: $(P+Q)+R = P+(Q+R)$ for all $P, Q, R \in \mathcal{Q}$
d. Identity: There is a $0 \in \mathcal{Q}$ such that $0 + P = P + 0 = P$
e. Inverse: For any $P \in \mathcal{Q}$ there exists a $(-P) \in \mathcal{Q}$ such that $P + (-P) = (-P) + P = 0$
Multiplication:
a. Closure: if $P, Q \in \mathcal{Q}$ then $PQ \in \mathcal{Q}$
b. Associativity: $(PQ)R = P(QR)$ for all $P, Q, R \in \mathcal{Q}$
c. Identity: There is a $1 \in \mathcal{Q}$ such that $1P = P1 = P$
d. Inverse: If $P \neq 0$, then there is a P^{-1} such that $PP^{-1} = P^{-1}P = 1$
2. Distributivity: $P(Q+R) = PQ+PR$ and $(Q+R)P = QP+RP$ for every $P, Q, R \in \mathcal{Q}$
3. No zero divisors: If $PQ = 0$, then either $P = 0$ or $Q = 0$.

2. Vectors as Quaternions

The fact that the symbols $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ are commonly used in vector analysis to represent elements of an orthonormal basis suggests that quaternions of the form $\beta\mathbf{i} + \gamma\mathbf{j} + \delta\mathbf{k}$

might be interpreted as vectors, and this is in fact the case. Moreover if two vectors

$$\mathbf{u} = \mathbf{u}_x\mathbf{i} + \mathbf{u}_y\mathbf{j} + \mathbf{u}_z\mathbf{k},$$
$$\mathbf{v} = \mathbf{v}_x\mathbf{i} + \mathbf{v}_y\mathbf{j} + \mathbf{v}_z\mathbf{k}$$

are multiplied as quaternions, the product is

$$\begin{aligned}\mathbf{uv} = & \; (-\mathbf{u}_x\mathbf{v}_x - \mathbf{u}_y\mathbf{v}_y - \mathbf{u}_z\mathbf{v}_z) \\ & + (\mathbf{u}_y\mathbf{v}_z - \mathbf{u}_z\mathbf{v}_y)\,\mathbf{i} \\ & + (\mathbf{u}_z\mathbf{v}_x - \mathbf{u}_x\mathbf{v}_z)\,\mathbf{j} \\ & + (\mathbf{u}_x\mathbf{v}_y - \mathbf{u}_y\mathbf{v}_x)\,\mathbf{k} \\ = & -(\mathbf{u}\cdot\mathbf{v}) + (\mathbf{u}\times\mathbf{v})\end{aligned} \tag{2}$$

where $\mathbf{u}\cdot\mathbf{v}$ and $\mathbf{u}\times\mathbf{v}$ are the familiar "dot product" and "cross product" of vector theory. Thus, dot and cross products, rather than being two separate forms of multiplication, are actually components of a single form of multiplication: quaternion multiplication.

Since $\mathbf{vu} = -\mathbf{v}\cdot\mathbf{u} + \mathbf{v}\times\mathbf{u}$, dot[2] and cross[3] products can be isolated as follows:

$$-\frac{\mathbf{uv}+\mathbf{vu}}{2} = \mathbf{u}\cdot\mathbf{v} \tag{3}$$

$$\frac{\mathbf{uv}-\mathbf{vu}}{2} = \mathbf{u}\times\mathbf{v} \tag{4}$$

We also obtain the length of a vector,

$$||\mathbf{v}|| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = (-\frac{\mathbf{vv}+\mathbf{vv}}{2})^{1/2} = \sqrt{-\mathbf{v}^2} \tag{5}$$

Thus, if $\mathbf{v}$ is a vector, then $\mathbf{v}/\sqrt{-\mathbf{v}^2}$ is a unit vector, and $\mathbf{n}$ is a unit vector if and only if $\mathbf{n}^2 = -1$.

3. Vector and Scalar Triple Products

Using the equality $(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = (\mathbf{v}\cdot\mathbf{w})\mathbf{u} + (\mathbf{u}\cdot\mathbf{w})\mathbf{v}$ and expansion 2 from the previous section, one can obtain the expansion

$$\begin{aligned}\mathbf{uvw} &= [-(\mathbf{u}\cdot\mathbf{v}) + (\mathbf{u}\times\mathbf{v})]\,\mathbf{w} \\ &= -(\mathbf{u}\cdot\mathbf{v})\mathbf{w} - (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} + (\mathbf{u}\times\mathbf{v})\times\mathbf{w} \\ &= -[\mathbf{u}\,\mathbf{v}\,\mathbf{w}] - (\mathbf{v}\cdot\mathbf{w})\mathbf{u} + (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}\end{aligned}$$

[2] Editor's note: If $P = p_0 + \mathbf{p}$ and $Q = q_0 + \mathbf{q}$, then define $P^* = p_0 - \mathbf{p}$ and $P\cdot Q = p_0q_0 + \mathbf{p}\cdot\mathbf{q}$. Then $P\cdot Q = \text{Scalar}(PQ^*) = (PQ^* + (PQ^*)^*)/2$.

[3] Editor's note: Using the notation of the previous footnote, $\mathbf{p}\times\mathbf{q} = (PQ - QP)/2$ — i.e. formula (4) ignores the scalar parts of P, Q.

where $[\mathbf{u}\,\mathbf{v}\,\mathbf{w}]$ represents the "scalar triple product"[4] $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$.

By considering different permutations of $\mathbf{u},\mathbf{v}$, and $\mathbf{w}$, one can isolate the scalar triple product[5] and vector triple product as follows:

$$\begin{aligned}[\mathbf{u}\,\mathbf{v}\,\mathbf{w}] &= \frac{(\mathbf{wvu}-\mathbf{uvw})}{2}\\ (\mathbf{u}\times\mathbf{v})\times\mathbf{w} &= \frac{(\mathbf{uvw}-\mathbf{wuv})}{2}\\ \mathbf{u}\times(\mathbf{v}\times\mathbf{w}) &= \frac{(\mathbf{uvw}-\mathbf{vwu})}{2}\end{aligned} \tag{6}$$

Thus, using quaternion notation, triple products are really no more difficult to represent than dot or cross products.

4. Representation of Rotation

The greatest strength of quaternions is their ability to represent rotation. In vector analysis, a rotation of angle θ about an axis $\mathbf{n}$ is represented by some matrix; for example, the rotation matrix for rotation by an angle θ around the $\mathbf{x}$-axis is:

$$(a_{ij}) = \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{bmatrix}$$

and the effect of applying this rotation to a vector $\mathbf{v}$ is given by matrix multiplication of (a_{ij}) by $\mathbf{v}$. The general matrix is very complicated and is given in books on computer graphics [4,5]. The matrix (a_{ij}) must be a "unitary matrix", which means that its columns, treated as vectors, are orthogonal and of unit length. Finding $\mathbf{n}$ and θ from (a_{ij}) involves finding the eigenvalues and eigenvectors of (a_{ij}) and can be rather awkward.

By contrast, in quaternion notation, the same rotation angle θ about axis $\mathbf{n}$ is represented by

$$\mathbf{v} \to R\mathbf{v}R^{-1}$$

[4]Editor's note: $[\mathbf{u}\,\mathbf{v}\,\mathbf{w}] = \begin{vmatrix}u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix}$.

[5]Editor's note: $[\mathbf{u}\,\mathbf{v}\,\mathbf{w}] = -\mathrm{Scalar}(\mathbf{uvw}) = -(\mathbf{uvw}+(\mathbf{uvw})^*)/2$. Following the previous footnotes, the notation $[\mathbf{p}\,\mathbf{q}\,\mathbf{r}]$ can be extended to include quaternions as follows:

$$[P\,Q\,R] = (\mathbf{p}\times\mathbf{q})\cdot\mathbf{r} = \left(\frac{PQ-QP}{2}\right)\cdot R = \mathrm{Scalar}\left(\frac{(PQ-QP)R^*}{2}\right).$$

Then triple products like the following make sense:

$$[P\mathbf{i}\,Q\mathbf{i}\,R\mathbf{i}] = \begin{vmatrix}p_0 & p_2 & p_3\\ q_0 & q_2 & q_3\\ r_0 & r_2 & r_3\end{vmatrix}.$$

where

$$R = (\cos\frac{\theta}{2}) + (\sin\frac{\theta}{2})\,\mathbf{n}. \tag{7}$$

The derivation of R, the explanation for the appearance of half-angles, and the proof that $R\mathbf{v}R^{-1}$ really is a vector can be found in many places [3,1]. It should be noted that:

1. It is much easier to retrieve the values of θ and $\mathbf{n}$, given R, than it is given the matrix (a_{ij}).

2. The vector $\mathbf{v}$ and the rotation R are represented by *the same kind of object*, namely quaternions. In vector theory, rotations are represented by matrices, a much different object than a vector. In quaternion theory, rotations themselves can be rotated!

5. Democracy of Unit Vectors, and Consequences

One of the most important features of quaternions is the fact that if $\mathbf{n}$ is a unit vector, then

$$\{\alpha + \beta\mathbf{n} : \alpha, \beta\,\text{real}\}$$

is isomorphic to the complex numbers. (This follows from the fact that $\mathbf{n}^2 = -1$.) This means that no unit vector is really any more important than any other unit vector. In a sense, the choice of $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ as coordinate bases is arbitrary; any mutually perpendicular (anti-commuting) unit vectors will do as well. This concept will be referred to as the "principle of democracy". This principle will be used to extend many concepts in complex numbers to apply to quaternions as well. In the following i is the imaginary number $\sqrt{-1}$.

One immediate consequence of this democracy is that any two quaternions of the forms $\alpha + \beta\mathbf{n}$ and $\gamma + \delta\mathbf{n}$ will commute under multiplication (after all, $\alpha + \beta i$ and $\gamma + \delta i$ commute.) Thus, although quaternions in general do not commute, certain classes of quaternions do. (Note that commutativity of multiplication is an *equivalence relation* among non-real quaternions.)

Another very important result is the following generalization of DeMoivres theorem:

Definition 1: $e^{\theta\mathbf{n}} = (\cos\theta) + (\sin\theta)\,\mathbf{n}$

Thus, a rotation of angle θ about axis $\mathbf{n}$ can also be represented as

$$R = e^{\theta\mathbf{n}/2} \tag{8}$$

In the same way, we can define trigonmetric and hyperbolic functions of quaternions in the same way as for complex numbers (e.g., since $\cos\theta i = \cosh\theta$, we have by democracy $\cos\theta\mathbf{n} = \cosh\theta$, for any angle θ and unit vector $\mathbf{n}$.)

Furthermore, since

$$\ln[e^r(\cos\theta + i\sin\theta)] = r + \theta i$$

then we should have

Definition 2: $\ln[e^r(\cos\theta + \mathbf{n}\sin\theta)] = r + \theta\mathbf{n}$

Here we should be careful in two respects: first we should always keep θ in the interval $(-\pi,\pi)$ to avoid ambiguity, and, secondly and more importantly, we must leave $\ln\alpha$ undefined for all $\alpha \leq 0$. After all, since $e^{\pi\mathbf{n}} = -1$ for every $\mathbf{n}$, every unit vector has a claim to the value of $\ln(-1)$, so $\ln(-1)$ will just have to stay undefined.

In any case, if P and Q commute, we can define

Definition 3: $P^Q = \exp[Q\ln P]$

Note that P and Q commute iff $(\ln P)$ and Q commute.

The following three relations hold for manipulating powers of quaternions:

1. $(PQ)^{-1} = Q^{-1}P^{-1}$.
2. $Q^{\alpha+\beta} = Q^{\alpha}Q^{\beta}$.
3. $Q^{\alpha\beta} = (Q^{\alpha})^{\beta}$ for $||Q|| \leq 1$ but in general, $e^{P+Q} \neq e^P e^Q$ and $e^{PQ} \neq (e^P)^Q$. Actually, $e^{P+Q} = e^P e^Q$ iff P and Q commute.
4. $e^{PQ} = (e^P)^Q$ if P and Q commute.

6. The Rotation $\sqrt{-\mathbf{vu}}$

Let $\mathbf{u}$ and $\mathbf{v}$ be unit vectors separated by an angle θ. Let g be the great circle containing $\mathbf{u}$ and $\mathbf{v}$, and let $\mathbf{n}$ be the pole of g, as shown in Figure 6.

Then,

$$\begin{aligned} -\mathbf{vu} &= \mathbf{v}\cdot\mathbf{u} - \mathbf{v}\times\mathbf{u} \\ &= \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\times\mathbf{v} \\ &= \cos\theta + \mathbf{n}\sin\theta \\ &= e^{\mathbf{n}\theta}. \end{aligned}$$

So

$$\sqrt{-\mathbf{vu}} = e^{\theta\mathbf{n}/2} \tag{9}$$

But $e^{\theta\mathbf{n}/2}$ is just the rotation with pole $\mathbf{n}$ that maps $\mathbf{u}$ into $\mathbf{v}$. Thus,

Theorem 4: If we want to rotate a sphere so that a unit vector $\mathbf{u}$ is shifted along a great circle until it reaches unit vector $\mathbf{v}$, the proper rotation is $\sqrt{-\mathbf{vu}}$.

7. The Rotation $[(\mathbf{wv}-\mathbf{vw})(\mathbf{wu}-\mathbf{uw})^{-1}]^{1/2}$

Suppose now that we wanted to rotate the unit sphere in such a way that $\mathbf{u}$ gets mapped onto $\mathbf{v}$, but a third point $\mathbf{w}$ gets mapped onto itself, as shown in Figure 7. What rotation should be used now? Well, if g is the great circle with pole $\mathbf{w}$ then $\mathbf{w}\times\mathbf{u}$ and $\mathbf{w}\times\mathbf{v}$ will both lie on g, and $\mathbf{w}\times\mathbf{u}$ will be mapped onto $\mathbf{w}\times\mathbf{v}$. Thus, the appropriate rotation is

$$\begin{aligned} [-(\mathbf{w}\times\mathbf{v})(\mathbf{w}\times\mathbf{u})]^{1/2} &= [(\mathbf{w}\times\mathbf{v})(\mathbf{w}\times\mathbf{u})^{-1}]^{1/2} \\ &= [(\frac{(\mathbf{wv}-\mathbf{vw})}{2})(\frac{(\mathbf{wu}-\mathbf{uw})}{2})^{-1}]^{1/2} \\ &= [(\mathbf{wv}-\mathbf{vw})(\mathbf{wu}-\mathbf{uw})^{-1}]^{1/2} \end{aligned}$$

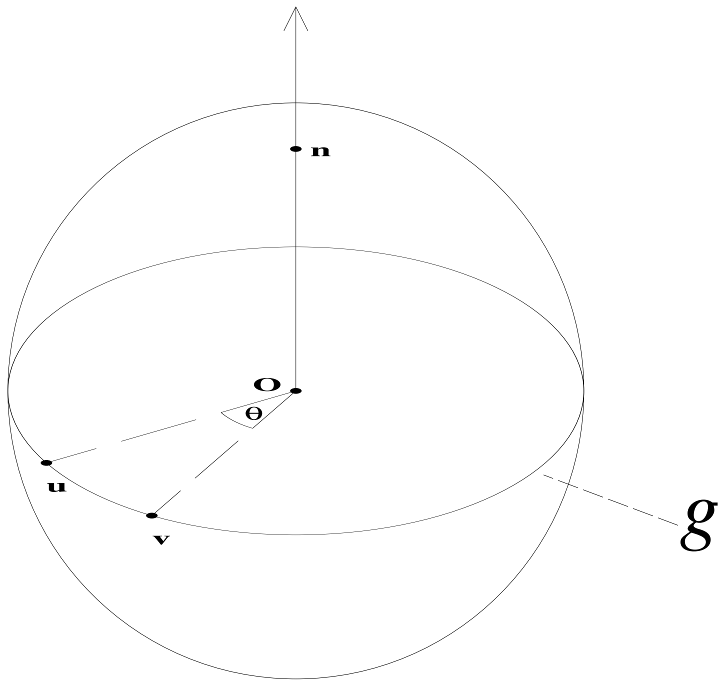

Figure 6: **u** is rotated into **v** along the great circle passing through them

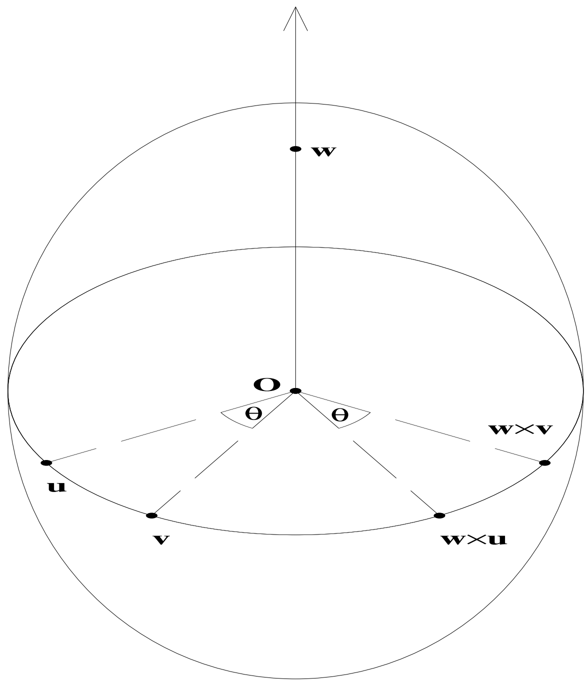

Figure 7: **u** rotates into **v**, while **w** is fixed

8. Reflections and Projections

We turn our attention now to reflections about, and projections onto, a line or plane. Let $\mathbf{n}$ be a unit vector. Then we can speak of

Definition 5: $Line(\mathbf{n}) = \{\mathbf{v} : \mathbf{nv} = \mathbf{vn}\}$

Definition 6: $Plane(\mathbf{n}) = \{\mathbf{v} : \mathbf{nv} = -\mathbf{vn}\}$

which are, respectively, the line passing through $\mathbf{0}$ and $\mathbf{n}$, and the plane passing through $\mathbf{0}$ perpendicular to $\mathbf{n}$.

Reflecting a vector across $Line(\mathbf{n})$ is the same as $180°$ rotation around the $\mathbf{n}$-axis, which is accomplished by

$$(\cos\frac{180°}{2}) + (\sin\frac{180°}{2})\,\mathbf{n} = \mathbf{n}. \qquad \text{(see Equation 8)}$$

Thus a vector $\mathbf{v}$ would be mapped onto the point $\mathbf{nvn}^{-1} = -\mathbf{nvn}$. If we consider Figure 8 we see that

Theorem 7: If $\mathbf{v}$ is a vector and $\mathbf{n}$ is a unit vector, then

1. The projection of $\mathbf{v}$ onto $Plane(\mathbf{n})$ is $\frac{\mathbf{v}+\mathbf{nvn}}{2}$.
2. The projection of $\mathbf{v}$ onto $Line(\mathbf{n})$ is $\frac{\mathbf{v}-\mathbf{nvn}}{2}$.
3. The reflection of $\mathbf{v}$ across $Plane(\mathbf{n})$ is $\mathbf{nvn}$.
4. The reflection of $\mathbf{v}$ across $Line(\mathbf{n})$ is $-\mathbf{nvn}$.

9. Affine Transformations

This section will describe two ways of representing affine transformations. The first method involves the formulas for representing reflections from Section 8. If $\mathbf{n}$ is a unit vector, then the mapping

$$\mathbf{v} \to \frac{(1+\alpha)\mathbf{v} + (1-\alpha)\mathbf{nvn}}{2} \tag{10}$$

"stretches" everything in the $\mathbf{n}$ directions by a factor of α, as shown in Figure 9. This can be seen by the fact that the right side of Equation 10 is a linear combination of $\mathbf{v}$ and $-\mathbf{nvn}$, made in such a way that if $\alpha = 1$ then $\mathbf{v}$ is mapped into $\mathbf{v}$, and if $\alpha = -1$ then $\mathbf{v}$ gets reflected into $-\mathbf{nvn}$.

Another form of affine transformation is the rotation

$$\mathbf{v} \to R\mathbf{v}R^{-1}$$

Presumably, every affine mapping should be expressible as the composition of rotations and stretchings like Equation 10, but in practice, this could become clumsy if too many of these rotations and stretchings are used in a row. There is a much nicer and more general way:

Theorem 8: The linear transformation with eigenvectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and real eigenvalues α, β, γ, is

$$\mathbf{v} \to \frac{\alpha[\mathbf{v}\,\mathbf{b}\,\mathbf{c}]\mathbf{a} + \beta[\mathbf{a}\,\mathbf{v}\,\mathbf{c}]\mathbf{b} + \gamma[\mathbf{a}\,\mathbf{b}\,\mathbf{v}]\mathbf{c}}{[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]}$$

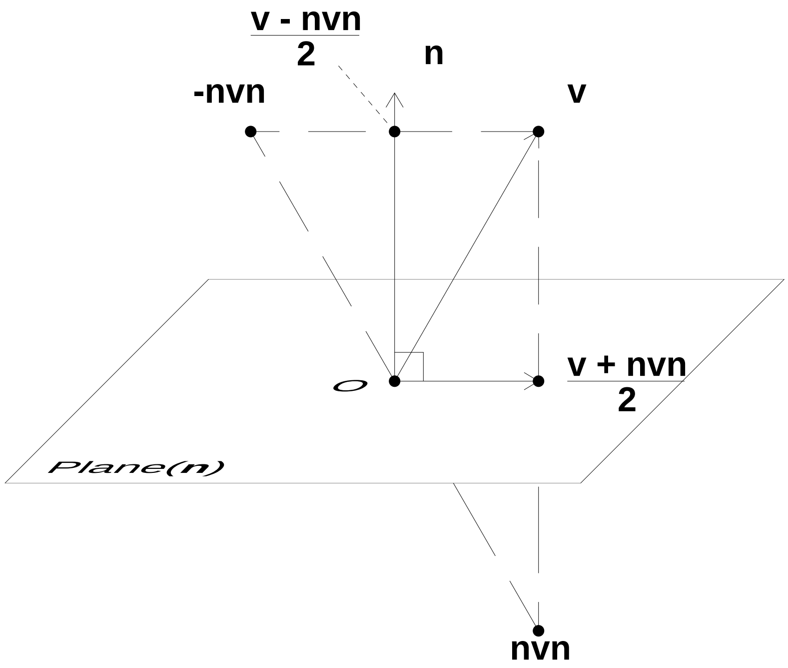

Figure 8: Relationship between **v**, its projection, and its reflection

Here $[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]$ and the like stand for the scalar triple product in Equation 6. It is easy to see that $\mathbf{a}$ is mapped into $\alpha\mathbf{a}$, $\mathbf{b}$ into $\beta\mathbf{b}$, and $\mathbf{c}$ into $\gamma\mathbf{c}$.[6] One can also show that Equations 8 and 10 are just special cases of Theorem 8.

10. Applications in computer vision

Most important computer vision functions can be represented simply using quaternions. We have already seen how to represent general rotations and affine transformations. This section develops expressions that are used exclusively in computer vision.

We define the image plane to be $Plane(\mathbf{v})$, the plane passing through the origin with surface normal $\mathbf{v}$. From Section 8 we may define the (parallel or orthogonal) projection of a point $\mathbf{p}$ onto $Plane(\mathbf{v})$ to be

$$\mathrm{pr}(\mathbf{p}) = \frac{\mathbf{p}+\mathbf{vpv}}{2}.$$

(Note that this is also a special case of Equation 10 with $\mathbf{a} = 0$.) Similarly we may define the (central or perspective) projection of a point $\mathbf{p}$ to be

$$\begin{aligned}\mathrm{PR}(\mathbf{p}) &= -\frac{\mathbf{p}+\mathbf{vpv}}{\mathbf{vp}+\mathbf{pv}}\\ &= \frac{\mathbf{v}\times(\mathbf{p}\times\mathbf{v})}{\mathbf{v}\cdot\mathbf{p}}.\end{aligned}$$

as shown in Figure 10.

Spherical projection onto a unit sphere can also be defined:

$$\mathrm{spr}(\mathbf{p}) = \mathbf{p}/\sqrt{-\mathbf{p}^2}$$

It was also mentioned in the last section that a general affine mapping can be represented as the composition of stretchings and rotations. However, if we are just studying

[6]Editor's note: It is also easy to see that Theorem 8 is "Cramer's Rule" in disguise (*Hint:* consider the determinant interpretation of the scalar triple products).

Theorem 8 can be extended to 4-dimensional transformations as follows. First, define

$$[P\,Q\,R\,S] = \begin{vmatrix} p_0 & p_1 & p_2 & p_3\\ q_0 & q_1 & q_2 & q_3\\ r_0 & r_1 & r_2 & r_3\\ s_0 & s_1 & s_2 & s_3\end{vmatrix}. \quad \text{Then expanding by cofactors,}$$

$$\begin{aligned}[P\,Q\,R\,S] &= P\cdot([Q\,R\,S]-[Q\mathbf{i}\,R\mathbf{i}\,S\mathbf{i}]\mathbf{i}-[Q\mathbf{j}\,R\mathbf{j}\,S\mathbf{j}]\mathbf{j}-[Q\mathbf{k}\,R\mathbf{k}\,S\mathbf{k}]\mathbf{k})\\ &= \mathrm{Scalar}(P([Q\,R\,S]+[Q\mathbf{i}\,R\mathbf{i}\,S\mathbf{i}]\mathbf{i}+[Q\mathbf{j}\,R\mathbf{j}\,S\mathbf{j}]\mathbf{j}+[Q\mathbf{k}\,R\mathbf{k}\,S\mathbf{k}]\mathbf{k})).\end{aligned}$$

Then the linear transformation with "eigenquaternions" A,B,C,D and real eigenvalues $\alpha,\beta,\gamma,\delta$ is

$$V \to \frac{\alpha[V\,B\,C\,D]A+\beta[A\,V\,C\,D]B+\gamma[A\,B\,V\,D]C+\delta[A\,B\,C\,V]D}{[A\,B\,C\,D]}$$

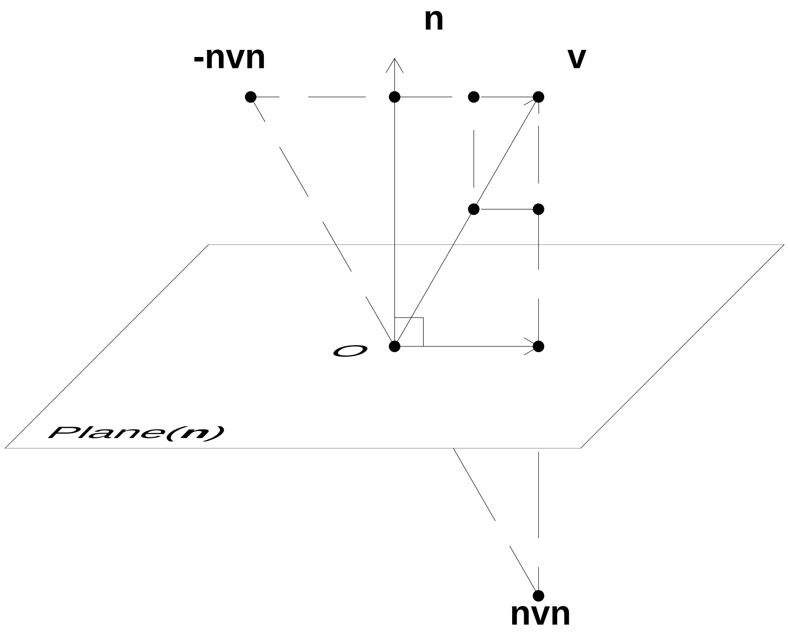

Figure 9: **v** is stretched by α in the direction of **n**

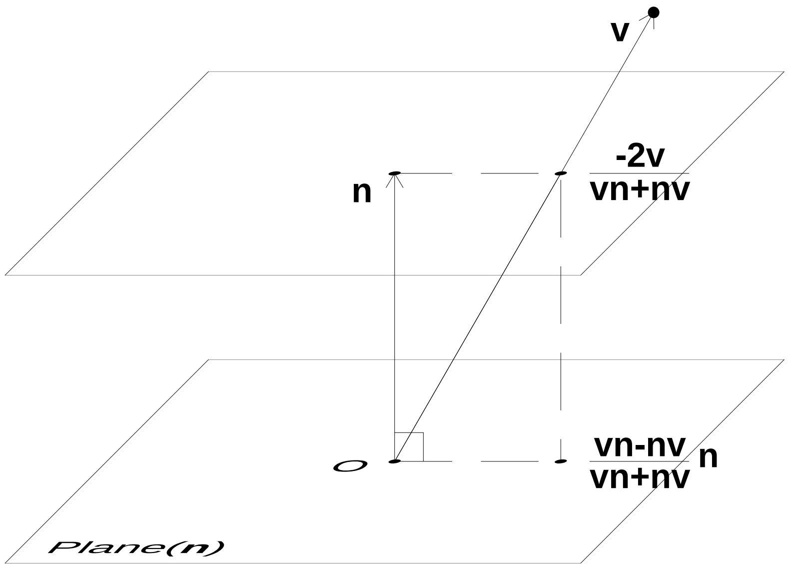

Figure 10: Parallel and central projection

a plane, all we nee are compositions of rotations and projections. In particular, consider the mapping

$$\mathbf{v} \rightarrow \frac{R\mathbf{v}R^{-1} + \mathbf{n}R\mathbf{v}R^{-1}\mathbf{n}}{2}$$

where R is some rotation $e^{\theta\mathbf{p}}$. This mapping will have the effect of rotating $\mathbf{v}$ by an angle θ about the axis $\mathbf{p}$, and then projecting it onto $Plane(\mathbf{n})$. If we allow R to be any quaternion, and not just a unit quaternion (a rotation), we can represent any affine transformation in this way, and can think of R as representing the affine transformation.

11. Describing the projection of the motion of a plane

Quaternions can be used to develop an interesting equation that relates motion of a plane in space to motion as seen on the image plane. This relationship is quite important in three-dimensional computer vision, since many objects are planar, or nearly so, over small areas. The relationship developed here is similar to the relationships developed by Kanade [2] using trigonometry, and Webb [6] using vectors and gradient space.

Consider a plane with surface normal $\mathbf{n}$. Let the plane rotate by some quaternion Q (we are ignoring the effects of translation here). Assume parallel projection. Under this assumption, the plane will be preserved to move by some affine transformation; let this transformation be represented by the quaternion A. Let the image plane be $Plane(\mathbf{v})$.

First consider the motion of the point in space. Let $\mathbf{y}$ be a point on the plane. The position of $\mathbf{y}$ after rotation is $Q\mathbf{y}Q^{-1}$. The position of this point on the image plane is $\frac{Q\mathbf{y}Q^{-1}+\mathbf{v}Q\mathbf{y}Q^{-1}\mathbf{v}}{2}$. Now consider the motion of the point on the image plane. The position of $\mathbf{y}$ before the motionis $\frac{\mathbf{y}+\mathbf{v}\mathbf{y}\mathbf{v}}{2}$. The affine transformation moves this point to

$$\frac{A\mathbf{y}A^{-1} + A\mathbf{v}\mathbf{y}\mathbf{v}A^{-1} + \mathbf{v}A\mathbf{y}A^{-1}\mathbf{v} + \mathbf{v}A\mathbf{v}\mathbf{y}\mathbf{v}A^{-1}\mathbf{v}}{4}$$

The observed image plane motion and the projection of the real motion must be the same, so that

$$\frac{Q\mathbf{y}Q^{-1} + \mathbf{v}Q\mathbf{y}Q^{-1}\mathbf{v}}{2} = \frac{A\mathbf{y}A^{-1} + A\mathbf{v}\mathbf{y}\mathbf{v}A^{-1} + \mathbf{v}A\mathbf{y}A^{-1}\mathbf{v} + \mathbf{v}A\mathbf{v}\mathbf{y}\mathbf{v}A^{-1}\mathbf{v}}{4}$$

The variable $\mathbf{y}$ in this equation is restricted to lie on the plane normal to $\mathbf{n}$. This restriction can be incorporated into the equation by writing $\mathbf{y} = \frac{\mathbf{x}+\mathbf{n}\mathbf{x}\mathbf{n}}{2}$, i.e., by writing $\mathbf{y}$ as the projection of some arbitrary quaternion $\mathbf{x}$. Once we do this substitution, we have an equation which is true for all quaternions. This equation can then be used to develop algorithms to determine motion in space from the observed affine transformation associated with motion.

12. Representation of Robot Arms

Another field in which quaternions should come in handy is the study of robot arm orientation. Traditionally a robot arm has been thought of as a series of links, each with

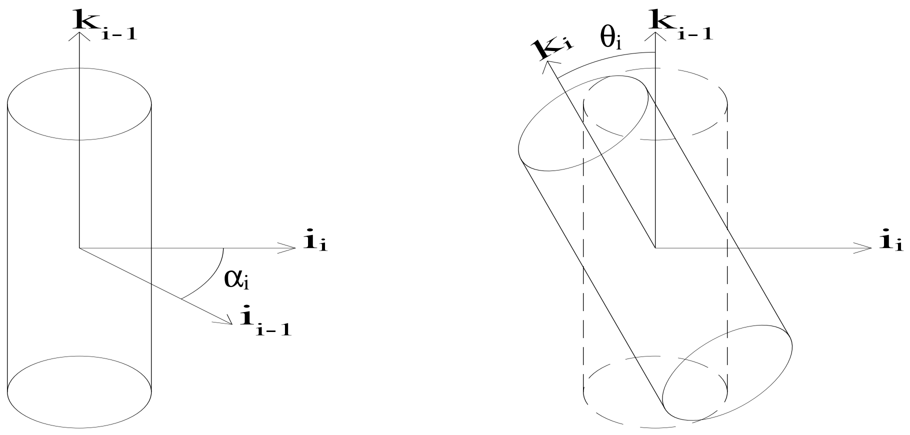

Figure 12: Coordinate system of a robot arm

its own coordinate system, as shown in Figure 12. The relation between successive links' coordinate systems is expressed in terms of a series of angles α_i and θ_i, and involves the rotation matrix

$$A^i_{i-1} = \begin{bmatrix} \cos\theta_i & -\cos\alpha_i \sin\theta_i & \sin\alpha_i \sin\theta_i \\ \sin\theta_i & \cos\alpha_i \cos\theta_i & -\sin\alpha_i \cos\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i \end{bmatrix}$$

But, recalling from Section 4 how much more elegantly rotations of coordinate systems can be expressed as quaternions, one is led to suspect that a quaternion representation of A^i_{i-1} should exist. In fact it is

$$R^i_{i-1} = e^{\theta_i \mathbf{i}/2} e^{\alpha_i \mathbf{k}/2}$$

These rotations are still composed

$$R^i_0 = R^1_0 R^2_1 ... R^i_{i-1}$$

The only important change is that if $\mathbf{v}_i$ represents a vector in link i coordinates, then its representation in link 0 coordinates is

$$\begin{aligned} \mathbf{v}_0 &= R^0_i \mathbf{v}_i (R^0_i)^{-1} \\ &= (R^i_0)^{-1} \mathbf{v}_i R^i_0 \end{aligned}$$

instead of

$$\begin{aligned} v_0 &= A^0_i v_i \\ &= (A^i_0)^T v_i \end{aligned}$$

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