Note on the General Quartic

M. E. Graber, A.M., Heidelberg University, Tiffin, Ohio.

American Mathematical Monthly, Vol. XVI (1909), 27-29.

The general quartic equation

$$\phi(x) \equiv a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0, \tag{1}$$

where the coefficients are all real, can be reduced to the form

$$\phi(x) \equiv ax^4 + bx^2 + c = 0, \tag{2}$$

by the following method.

Suppose the quartic resolved into the factors:

$$(a_1x^2 + b_1x + c_1) \qquad \text{and} \qquad (a_2x^2 + b_2x + c_2).$$

Then effect the rational bilinear transformation $x = \frac{\alpha y+\beta}{y+1}$, obtaining

$$\phi(x) = \left[\frac{a_1(\alpha y+\beta)^2}{(y+1)^2} + \frac{b_1(\alpha y+\beta)}{y+1} + c_1\right]\left[\frac{a_2(\alpha y+\beta)^2}{(y+1)^2} + \frac{b_2(\alpha y+\beta)}{y+1} + c_2\right] = 0.$$

In this expression α and β may be chosen so that the coefficients of the first powers of y shall be zero, after clearing of fractions.

The values of α and β fulfilling this condition are easily found to depend upon

$$\alpha\beta = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \tag{3}$$

$$\alpha + \beta = 2\frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}. \tag{4}$$

That α and β are real, is determined from the equation

$$t^2 + 2\frac{(a_2c_1 - a_1c_2)}{(a_1b_2 - a_2b_1)}t + \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} = 0, \tag{5}$$

by means of the discriminant relation

$$
\begin{aligned}
(6)\qquad \triangle &\equiv \left[\frac{2(a_2c_1-a_1c_2)}{(a_1b_2-a_2b_1)}\right]^2-4\left[\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}\right]\\
&=\frac{4\left[(a_2c_1-a_1c_2)^2-(b_1c_2-b_2c_1)(a_1b_2-a_2b_1)\right]}{(a_1b_2-a_2b_1)^2}.
\end{aligned}
$$

If v_1, w_1, and v_2, w_2 are the two pairs of roots of the quartic, we may assume, without loss of generality, that v_1 is greater than either v_2 or w_2 and w_1 is greater than either v_2 or w_2. From the two quadratic factors whose roots are, respectively, v_1, w_1, and v_2, w_2, we obtain

$$
(7)\qquad -\frac{b_1}{a_1}=v_1+w_1,\quad \frac{c_1}{a_1}=v_1w_1,\quad -\frac{b_2}{a_2}=v_2+w_2,\quad \text{and}\quad \frac{c_2}{a_2}=v_2w_2,
$$

and these values substituted in the discriminant give for the determinations of its signs,

$$
\begin{aligned}
(8)\qquad & a_1^2a_2^2(v_1-v_2)(v_1-w_2)(w_1-v_2)(w_1-w_2)\\
&=(a_2c_1-a_1c_2)^2-(b_1c_2-b_2c_1)(a_1b_2-a_2b_1).
\end{aligned}
$$

If all the roots are real it is evident from (8) that the discriminant is positive. Likewise, if two roots are real and two imaginary, and also if all four roots are imaginary, the discriminant is seen to be positive, the conjugate pairs being, v_1, w_1, and v_2, w_2. Hence, (5) has real roots, and consequently α and β are real.

In the exceptional case where $\frac{b_1}{a_1}=\frac{b_2}{a_2}$, the substitution

$$
x=y-\frac{b_1}{2a_1}=y-\frac{b_2}{2a_2},
$$

gives directly $\phi(x)=A_1y^4+A_2y^2+A_3$, where A_1, A_2, A_3 are real constants.

Finally, from (3), (4) and (7),

$$
(9)\qquad \alpha\beta=\frac{(v_2+w_2)v_1w_1-v_2w_2(v_1+w_1)}{(v_1+w_1)-(v_2+w_2)},
$$

$$
(10)\qquad \text{and}\quad \alpha+\beta=\frac{2v_1w_1-2v_2w_2}{(v_1+w_1)-(v_2+w_2)}.
$$

From these equations it is easily seen that

$$
\begin{aligned}
2\alpha\beta-(\alpha+\beta)(v_1+w_1)&=-2v_1w_1,\\
\text{and}\quad 2\alpha\beta-(\alpha+\beta)(v_2+w_2)&=-2v_2w_2.
\end{aligned}
$$

But these are the characteristic equations for harmonic pairs. Therefore, the pair α, β is harmonic with respect to the pairs v_1, w_1 and v_2, w_2.