ON MULTIPLE ALGEBRA

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It has been said that "the human mind has never invented a labor- saving machine equal to algebra."[1] If this be true, it is but natural and proper that an age like our own, characterized by the multiplication of labor-saving machinery, should be distinguished by an unexampled development of this most refined and most beautiful of machines. That such has been the case, none will question. The improvement has been in every part. Even to enumerate the principal lines of advance would be a task for any one; for me an impossibility. But if we should ask, in what direction the advance has been made which is to characterize the development of algebra in our day, we may, I think, point to that broadening of its field and methods which gives us *multiple algebra*.

Of the importance of this change in the conception of the office of algebra, it is hardly necessary to speak: that it is really characteristic of our time will be most evident if we go back some two or three-score years, to the time when the seeds were sown which are now yielding so abundant a harvest. The failure of Möbius, Hamilton, Grassmann, Saint-Venant to make an immediate impression upon the course of mathematical thought in any way commensurate with the importance of their discoveries is the most conspicuous evidence that the times were not ripe for the methods which they sought to introduce. A satisfactory theory of the imaginary quantities of ordinary algebra, which is essentially a simple case of multiple algebra, with difficulty obtained recognition in the first third of this century. We must observe that this *double algebra*, as it has been called, was not sought for or invented;—it forced itself, unbidden, upon the attention of mathematicians, and with its rules already formed.

But the idea of double algebra, once received, although as it is were unwillingly, must have suggested to many minds more or less distinctly the possibility of

[1] *The Nation*, vol. xxxiii, p. 237.

other multiple algebras, of higher orders, possessing interesting or useful properties.

The application of double algebra to the geometry of the plane suggested not unnaturally to Hamilton the idea of a triple algebra which should be capable of a similar application to the geometry of three dimensions. He was unable to find a satisfactory triple algebra, but discovered at length a quadruple algebra, *quaternions*, which answered his purpose, thus satisfying, as he says in one of his letters, an intellectual want which had haunted him at least fifteen years. So confident was he of the value of this algebra, that the same hour he obtained permission to lay his discovery before the Royal Irish Academy, which he did on November 13, 1843.[2] This system of multiple algebra is far better known than any other, except the ordinary double algebra of imaginary quantities,—far too well known to require any especial notice at my hands. All that here requires our attention is the close historical connection between the imaginaries of ordinary algebra and Hamilton's system, a fact emphasized by Hamilton himself and most writers on quaternions. It was quite otherwise with Möbius and Grassmann.

The point of departure of the *Barycentrischer Calcul* of Möbius, published in 1827,—a work of which Clebsch has said that it can never be admired enough,[3] —is the use of equations in which the terms consist of letters representing points with numerical coefficients, to express barycentric relations between the points. Thus, that the point S is the centre of gravity of weights, a,b,c,d, placed at the points A,B,C,D, respectively, is expressed by the equation

$$(a+b+c+d)S = aA+bB+cC+dD.$$

An equation of the more general form

$$aA+bB+cC+\text{etc.} = pP+qQ+rR+\text{etc.}$$

signifies that the weights a,b,c, etc., at the points A,B,C, etc., have the same sum and the same centre of gravity as the weights p,q,r, etc., at the points P,Q,R, etc., or, in other words, that the former are barycentrically equivalent to the latter. Such equations, of which each represents four ordinary equations, may evidently be multiplied or divided by scalars,[4] may be added or subtracted, and may have

[2] *Phil. Mag.* (3), vol. xxv, p. 490; *North British Review*, vol. xlv (1866), p. 57.

[3] See his eulogy on Plücker, p. 14, *Gött. Abhandl.*, vol. xvi.

[4] I use this term in Hamilton's sense, to denote the ordinary positive and negative quantities of algebra. It may, however, be observed that in most cases in which I shall have occasion to use it, the proposition would hold without exclusion of imaginary quantities,—that this exclusion is generally for simplicity and not from necessity.

their terms arranged and transposed, exactly like the ordinary equations of algebra. It follows that the elimination of letters representing points from equations of this kind is performed by the rules of ordinary algebra. This is evidently the beginning of a quadruple algebra, and is identical, as far as it goes, with Grassmann's marvellous geometrical algebra.

In the same work we find also, for the first time so far as I am aware, the distinction of positive and negative consistently carried out on the designation of segments of lines, of triangles, and of tetrahedra, viz., that a change in place of two letters, in such expressions as $AB, ABC, ABCD$, is equivalent to prefixing the negative sign. It is impossible to overestimate the importance of this step, which gives to designations of this kind the generality and precision of algebra.

Moreover, if A, B, C are three points in the same straight line, and D any point outside of that line, the author observes that we have

$$AB + BC + CA = 0.$$

and also, with D prefixed,

$$DAB + DBC + DCA = 0.$$

Again, if A, B, C, D are four points in the same plane, and E any point outside of that plane, we have

$$ABC - BCD + CDA - DAB = 0,$$

and also, with E prefixed,

$$EABC - EBCD + ECDA - EDAB = 0.$$

The similarity to multiplication in the derivation of these formulæcannot have escaped the author's notice. Yet he does not seem to have been able to generalize these processes. It was reserved for the genius of Grassmann to see that AB might be regarded as the product of A and B. DAB as the product of D and AB, and $EABC$ as the product of E and ABC. That Möbius could not make this step was evidently due to the fact that he had not the conception of the addition of other multiple quantities than such as may be represented by masses situated at points. Even the addition of vectors (i.e., the fact that the composition of directed lines could be treated as an addition) seems to have been unknown to him at this time, although he subsequently discovered it, and used it in his *Mechanik des Himmels*, which was published in 1843. This addition of vectors, or *geometrical addition*, seems to have occurred independently to many persons.

Seventeen years after the *Barycentrischer Calcul*, in 1844, the year in which Hamilton's first papers on quaternions appeared in print, Grassmann published his *Lineale Ausdehnungslehre*, in which he developed the idea and the properties of the *external* or *combinatorial product*, a conception which is perhaps to be regarded as the greatest monument of the author's genius. This volume was to have been followed by another, of the nature of which some intimation was given in the preface and in the work itself. We are especially told that the *internal product*,[5] which for vectors is identical except in sign with the scalar part of Hamilton's product (just as Grassmann's external product of two vectors is practically identical with the vector part of Hamilton's product), and the *open product*,[6] which in the language of to-day would be called a matrix, were to be treated in the second volume. But both the internal product of vectors and the open product are clearly defined, and their fundamental properties indicated, in this first volume.

This remarkable work remained unnoticed for more than twenty years, a fact which was doubtless due in part to the very abstract and philosophical manner in which the subject was presented. In consequence of this neglect the author changed his plan, and instead of a supplementary volume published in 1862 a single volume entitled *Ausdehnungslehre*, in which were treated, in an entirely different style, the same topics as in the first volume, as well as those which he had reserved for the second.

Deferring for the moment the discussion of these topics in order to follow the course of events, we find in the year following the first *Ausdehnungslehre* a remarkable memoir of Saint-Venant,[7] in which are clearly described the addition both of vectors and of oriented areas, the differentiation of these with respect to a scalar quantity, and a multiplication of two vectors and of a vector and an oriented area. These multiplications, called by the author *geometrical*, are entirely identical with Grassmann's external multiplication of the same quantities.

It is a striking fact in the history of the subject, that the short period of less than two years was marked by the appearance of well-developed and valuable systems of multiple algebra by British, German, and French authors, working apparently entirely independently of one another. No system of multiple algebra had appeared before, so far as I know, except such as were confined to additive processes with multiplication by scalars, or related to the ordinary double algebra of imaginary quantities. But the appearance of a single one of these systems would have been sufficient to mark an epoch, perhaps the most important epoch in the history of the

[5]See the preface.

[6]See §172.

[7]*Comptes Rendus*, vol. xxi, p. 620.

subject.

In 1853 and 1854, Cauchy published several memoirs on what he called *clefs algébriques*.[8] These were units subject generally to combinatorial multiplication. His principal application was to the theory of elimination. In this application, as in the law of multiplication, he had been anticipated by Grassmann.

We come next to Cayley's celebrated *Memoir on the Theory of Matrices*[9] in 1858, of which Sylvester has said that it seems to him to have ushered in the reign of Algebra the Second.[10] I quote this dictum of a master as showing his opinion of the importance of the subject and of the memoir. But the foundations of the theory of matrices, regarded as multiple quantities, seem to me to have been already laid in the *Ausdehnungslehre* of 1844. To Grassmann's treatment of this subject we shall recur later.

After the *Ausdehnungslehre* of 1862, already mentioned, we come to Hankel's *Vorlesungen über die complexen Zahlen*, 1867. Under this title the author treats of the imaginary quantities of ordinary algebra, of what he calls *alternirende Zahlen*, and of quaternions. These alternate numbers, like Cauchy's *clefs*, are quantities subject to Grassmann's law of combinatorial multiplication. This treatise, published twenty-three years after the first *Ausdehnungslehre*, marks the first impression which we can discover of Grassmann's ideas upon the course of mathematical thought. The transcendent importance of these ideas was fully appreciated by the author, whose very able work seems to have had considerable influence in calling the attention of mathematicians to the subject.

In 1870, Professor Benjamin Peirce published his *Linear Associative Algebra*, subsequently developed and enriched by his son, Professor C. S. Peirce. The fact that the edition was lithographed seems to indicate that even at this late date a work of this kind could only be regarded as addressed to a limited number of readers. But the increasing interest in such subjects is shown by the republication of this memoir in 1881,[11] as by that of the first *Ausdehnungslehre* in 1878.

The article on quaternions which has just appeared in the *Encyclopædia Britannica* mentions twelve treatises, including second editions and translations, besides the original treatises of Hamilton. That all the twelve are later than 1861 and all but two later than 1872 shows the rapid increase of interest in this subject in the last years.

Finally, we arrive at the *Lectures on the Principles of Universal Algebra* by the distinguished foreigner whose sojourn among us has given such an impulse to

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[8] *Comptes Rendus*, vols. xxxvi, ff.

[9] *Phil. Trans.*, vol. cxlviii.

[10] *Amer. Journ. Math.*, vol. vi, p. 271.

[11] *Amer. Journ. Math.*, vol. iv.

mathematical study in this country. The publication of these lectures, commenced in 1884 in the *American Journal of Mathematics*, has not as yet been completed,—a want but imperfectly supplied by the author's somewhat desultory publication of many remarkable papers on the same subject (which might be more definitely expressed as the algebra of matrices) in various foreign journals.

It is not an accident that this century has seen the rise of multiple algebra. The course of the development of ideas in algebra and in geometry, although in the main independent of any aid from this source, has nevertheless to a very large extent been of a character which can only find its natural expression in multiple algebra.

Our Modern Higher Algebra is especially occupied with the theory of linear transformations. Now what are the first notions which we meet in this theory? We have a set of n variables, say x, y, z, and another set, say x', y', z', which are homogeneous linear functions of the first, and therefore expressible in terms of them by means of a block of n^2 coefficients. Here the quantities occur by sets, and invite the notations of multiple algebra. It was in fact shown by Grassmann in his first *Ausdehnungslehre* and by Cauchy nine years later, that the notations of multiple algebra afford a natural key to the subject of elimination.

Now I do not merely mean that we may save a little time or space by writing perhaps ρ for x, y and z; ρ' for x', y' and z'; and Φ for a block of n^2 quantities. But I mean that the subject as usually treated under the title of determinants has a stunted and misdirected development on account of the limitations of single algebra. This will appear from a very simple illustration. After a little preliminary matter the student comes generally to a chapter entitled "Multiplication of Determinants," in which he is taught that the product of the determinants of two matrices may be found by performing a somewhat lengthy operation on the two matrices, by which he obtains a third matrix, and then taking the determinant of this. But what significance, what value has this theorem? For aught that appears in the majority of treatises which I have seen, we have only a complicated and lengthy way of performing a simple operation. The real facts of the case may he stated as follows:

Suppose the set of n quantities ρ' to be derived from the set ρ by the matrix Φ, which we may express by

$$\rho' = \Phi.\rho;$$

and suppose the set ρ'' to be derived from the set ρ' by the matrix Ψ, i.e.,

$$\rho'' = \Psi.\rho'.$$

and

$$\rho'' = \Psi.\Phi.\rho;$$

it is evident that ρ'' can be derived from ρ by the operation of a single matrix, say Θ, i.e.,

$$\rho'' = \Theta.\rho,$$

so that

$$\Theta = \Psi.\Phi.$$

In the language of multiple algebra Θ is called the product of Ψ and Φ. It is of course interesting to see how it is derived from the latter, and it is little more than a schoolboy's exercise to determine this. Now this matrix Θ has the property that its determinant is equal to the products of the determinants of Ψ and Φ. And this property is all that is generally stated in the books, and the fundamental property, which is all that gives the subject its interest, that Θ is itself the product of Ψ and Φ in the language of multiple algebra, i.e., that operating by Θ is equivalent to operating successively by Φ and Ψ, is generally omitted. The chapter on this subject, in most treatises which I have seen, reads very like the play of Hamlet with Hamlet's part left out.

And what is the cause of this omission? Certainly not ignorance of the property in question. The fact that it is occasionally given would be a sufficient bar to this answer. It is because the author fails to see that his real subject is matrices and not determinants. Of course, in a certain sense, the author has a right to choose his subject. But this does not mean that the choice is unimportant, or that it should be determined by chance or by caprice. The problem well put is half solved, as we all know. If one chooses the subject ill, it will develop itself in a cramped manner.

But the case is really much worse than I have stated it. Not only is the true significance of the formation of Θ from Ψ and Φ not given, but the student is often not taught to form the matrix which is the product of Ψ and Φ, but one which is the product of one of these matrices and the conjugate of the other. Thus the proposition which is proved loses all its simplicity and significance, and must be recast before the instructor can explain its true bearings to the student. This fault has been denounced by Sylvester, and if anyone thinks I make too much of the standpoint from which the subject is viewed, I will refer him to the opening paragraphs of the "Lectures on Universal Algebra" in the sixth volume of the *American Journal of Mathematics*, where, with a wealth of illustration and an energy of diction which I cannot emulate, the most eloquent of mathematicians expresses his sense of the importance of the substitution of the idea of the matrix for that of the determinant. If then so important, why was the idea of the matrix let slip? Of course the writers on this subject had it to commence with. One cannot even define a determinant without the idea of a matrix. The simple fact is that in general the writers on this subject have especially developed those ideas which are naturally expressed in simple algebra, and have postponed or slurred over or omitted altogether those ideas which find their natural expression in multiple algebra. But in this subject the latter happen to be the fundamental ideas, and those which ought to direct the whole course of thought.

I have taken a very simple illustration, perhaps the very first theorem which meets the student after those immediately connected with the introductory definitions, both because the simplest illustration is really the best, and because I am here most at home. But the principles of multiple algebra seem to me to shed a flood of light into every corner of the subjects usually treated under the title of determinants, the subject gaining as much in breadth from the new notions as in simplicity from the new notations; and in the more intricate subjects of invariants, covariants, etc., I believe that the principles of multiple algebra are ready to perform an equal service. Certainly they make many things seem very simple to me which I should otherwise find difficult of comprehension.

Let us turn to geometry.

If we were asked to characterize in a single word our modern geometry, we would perhaps say that it is a geometry of position. Now position is essentially a multiple quantity, or, if you prefer, is naturally represented in algebra by a multiple quantity. And the growth in this century of the so-called synthetic as opposed to analytical geometry seems due to the fact that by the ordinary analysis geometers could not easily express, except in a cumbersome and unnatural manner, the sort of relations in which they were particularly interested. With the introduction of the notations of multiple algebra, this difficulty falls away, and with it the opposition between synthetic and analytical geometry.

It is, however, interesting and very instructive to observe how the ingenuity of mathematicians has often triumphed over the limitations of ordinary algebra. A conspicuous example and one of the simplest is seen in the *Mécanique Analytique*, where the author, by the use of what are sometimes called indeterminate equations, is able to write in one equation the equivalent of an indefinite number. Thus the equation

$$Xdx + Ydy + Zdz = 0,$$

by the indeterminateness of the values of dx, dy, dz, is made equivalent to the three equations

$$X = 0, Y = 0, Z = 0.$$

It is instructive to compare this with

$$Xi + Yj + Zk = 0,$$

which is the form that Hamilton or Grassmann would have used. The use of this analytical artifice, if such it can be called, runs all through the work and is fairly characteristic of it.

Again, the introduction of the potential in the theory of gravity, or electricity, or magnetism, gives us a scalar quantity instead of a vector as the subject of study; and in mechanics generally the use of the force-function substitutes a simple quantity for a complex. This method is in reality not different from that just mentioned, since Lagrange's indeterminate equation expresses, at least in its origin, the variation of the force-function. It is indeed the real beauty of Lagrange's method that it is not so much an analytical artifice, as the natural development of the subject.

In modern analytical geometry we find methods in use which are exceedingly ingenious, and give forms curiously like those of multiple algebra, but which, at least if logically carried out very far, are excessively artificial, and that for the expression of the simplest things. The simplest conceptions of the geometry of three dimensions are points and planes, and the simplest relation between these is that a point lies in a plane. Let us see how these notions have been handled by means of ordinary algebra, and by multiple algebra. It will illustrate the characteristic difference of the methods, perhaps as well as the reading of an elaborate treatise.

In multiple algebra a point is designated by a single letter, just as it is in what is called synthetic geometry, and as it generally is by the ordinary analyst when he is not writing equations. But in his equations, instead of a single letter the analyst introduces several letters (coordinates) to represent the point.

A plane may be represented in multiple algebra as in synthetic geometry by a single letter; in the ordinary algebra it is sometimes represented by three coordinates, for which it is most convenient to take the reciprocals of the segments cut off by the plane on three axes. But the modern analyst has a more ingenious method of representing the plane. He observes that the equation of the plane may be written

$$\xi x + \eta y + \zeta z = 1, \tag{1}$$

where ξ, η, ζ are the reciprocals of the segments, and x, y, z are the coordinates of any point in the plane. Now if we set

$$p = \xi x + \eta y + \zeta z, \tag{2}$$

this letter will represent an expression which represents the plane. In fact, we may say that p implicitly contains ξ, η, and ζ, which are the coordinates of the plane. We may therefore speak of the plane p, and for many purposes can introduce the letter p into our equations instead of ξ, η, ζ. For example, the equation

$$p''' = \frac{p' + p''}{2} \tag{3}$$

is equivalent to the three equations

$$\xi''' = \frac{\xi' + \xi''}{2}, \eta''' = \frac{\eta' + \eta''}{2}, \zeta''' = \frac{\zeta' + \zeta''}{2}. \tag{4}$$

It is to be noticed that on account of the indeterminateness of the x, y, and z, this method, regarded as an analytical artifice, is identical with that of Lagrange, also that in multiple algebra we should have an equation of precisely the same form as (3) to express the same relation between the planes, but that the equation would be explained to the student in a totally different manner. This we shall see more particularly hereafter.

It is curious that we have thus a simpler notation for a plane than for a point. This however may be reversed. If we commence with the notion of the coordinates of a plane, ξ, η, ζ, the equation of a point (i.e., the equation between ξ, η, ζ which will hold for every plane passing through the point) will be

$$x\xi + y\eta + z\zeta = 1, \tag{5}$$

where x, y, z are the coordinates of the point. Now if we set

$$q = x\xi + y\eta + z\zeta, \tag{6}$$

we may regard the single letter q as representing the point, and use it, in many cases, instead of the coordinates x, y, z, which indeed it implicitly contains. Thus we may write

$$q''' = \frac{q' + q''}{2} \tag{7}$$

for the three equations

$$x''' = \frac{x' + x''}{2}, y''' = \frac{y' + y''}{2}, z''' = \frac{z' + z''}{2}. \tag{8}$$

Here, by an analytical artifice, we come to equations identical in form and meaning with those used by Hamilton, Grassmann, and even by Möbius in 1827. But the explanations of the formulæwould differ widely. The methods of the founders of multiple algebra are characterized by a bold simplicity, that of the modern geometry by a somewhat bewildering ingenuity. That p and q represent the same expression (in one case x, y, z, and in the other ξ, η, ζ being indeterminate) is a circumstance which may easily become perplexing. I am not quite certain that it would be convenient to use both of these abridged notations at the same time. In fact, if the geometer using these methods were asked to express by an equation in p

and q that the point q lies in the plane p. he might find himself somewhat entangled in the meshes of his own ingenuity, and need some new artifice to extricate himself. I do not mean that his genius might not possibly be equal to the occasion, but I do mean very seriously that it is a vicious method which requires any ingenuity or any artifice to express so simple a relation.

If we use the methods of multiple algebra which are most comparable to those just described, a point is naturally represented by a vector (ρ) drawn to it from the origin, a plane by a vector (σ) drawn from the origin perpendicularly toward the plane and in length equal to the reciprocal of the distance of the plane from the origin. The equation

$$\sigma''' = \frac{\sigma' + \sigma''}{2} \tag{9}$$

will have precisely the same meaning as equation (3), and

$$\rho''' = \frac{\rho' + \rho''}{2} \tag{10}$$

will have precisely the same meaning as equation (7), viz., that the point ρ''' is in the middle between ρ' and ρ''. That the point ρ lies in the plane σ is expressed by equating to unity the product of ρ and σ called by Grassmann internal, or by Hamilton called the scalar part of the product taken negatively. By whatever name called, the quantity in question is the product of the lengths of the vectors and the cosine of the included angle. It is of course immaterial what particular sign we use to express this product, as whether we write

$$\rho.\sigma = 1, \text{ or } S\rho\sigma = -1. \tag{11}$$

I should myself prefer the simplest possible sign for so simple a relation. It may be observed that ρ and σ may be expressed as the geometrical sum of their components parallel to a set of perpendicular axes, viz..

$$\rho = xi + yj + zk, \sigma = \xi i + \eta j + \zeta k. \tag{12}$$

By substitution of these values, equation (11) becomes by the laws of this kind of multiplication

$$x\xi + y\eta + z\zeta = 1. \tag{13}$$

My object in going over these elementary matters is to call attention to the very roundabout way in which the ordinary analysis makes out to represent a point or a plane by a single letter, as distinguished from the directness aud simplicity of the notations of multiple algebra, and also to the fact that the representations of points and planes by single letters in the ordinary analysis are not, when obtained, as amenable to analytical treatment as are the notations of multiple algebra.

I have compared that form of the ordinary analysis which relates to Cartesian axes with a vector analysis. But the case is essentially the same if we compare the form of ordinary analysis which relates to a fundamental tetrahedron with Grassmann's geometrical analysis, founded on the point as the elementary quantity.

In the method of ordinary analysis a point is represented by four coordinates, of which each represents the distance of the point from a plane of the tetrahedron divided by the distance of the opposite vertex from the same plane. The equation of a plane may be put in the form

$$\xi x+\eta y+\zeta z+\omega w=0, \tag{14}$$

where ξ,η,ζ,ω are the distances of the plane from the four points, and x,y,z,w are the coordinates of any point in the plane. Here we may set

$$p=\xi x+\eta y+\zeta z+\omega w, \tag{15}$$

and say that p represents the plane. To some extent we can introduce this letter into equations instead of ξ,η,ζ,ω. Thus the equation

$$lp'+mp''+np'''=0 \tag{16}$$

(which denotes that the planes p',p'',p''', meet in a common line, making angles of which the sines are proportional to l,m, and n) is equivalent to the four equations

$$l\xi'+m\xi''+n\xi'''=0, l\eta'+m\eta''+n\eta'''=0, \text{etc.} \tag{17}$$

Again, we may regard ξ,η,ζ,ω as the coordinates of a plane. The equation of a point will then be

$$x\xi+y\eta+z\zeta+w\omega=0. \tag{18}$$

If we set

$$q=x\xi+y\eta+z\zeta+w\omega, \tag{19}$$

we may say that q represents the point. The equation

$$q'''=\frac{q'+q''}{2}, \tag{20}$$

which indicates that the point q''' bisects the line between q' and q'', is equivalent to the four equations

$$\xi'=\frac{\xi''+\xi'''}{2}, \xi'=\frac{\xi''+\xi'''}{2}, \text{etc.} \tag{21}$$

To express that the point q lies in the plane p does not seem easy, without going back to the use of coordinates.

The form of multiple algebra which is to be compared to this is the geometrical algebra of Möbius and Grassmann, in which points without reference to any origin are represented by single letters, say by Italic capitals, and planes may also be represented by single letters, say by Greek capitals. An equation like

$$Q''' = \frac{Q' + Q''}{2}, \tag{22}$$

has exactly the same meaning as equation (20) of ordinary algebra. So

$$l\Pi' + m\Pi'' + n\Pi''' = 0 \tag{23}$$

has precisely the same meaning as equation (16) of ordinary algebra. That the point Q lies in the plane Π is expressed by equating to zero the product of Q and Π which is called by Grassmann external and which might be defined as the distance of the point from the plane. We may write this

$$Q \times \Pi = 0. \tag{24}$$

To show that so simple an expression is really amenable to analytical treatment, I observe that Q may be expressed in terms of any four points (not in the same plane) on the barycentric principle explained above, viz.,

$$Q = xA + yB + zC + wD, \tag{25}$$

and Π may be expressed in terms of combinatorial products of A, B, C, and D, viz.,

$$\Pi = \xi B \times C \times D + \eta C \times A \times D + \zeta D \times A \times B + \omega A \times C \times B, \tag{26}$$

and by these substitutions, by the laws of the combinatorial product to be mentioned hereafter, equation (24) is transformed into

$$w\omega + x\xi + y\eta + z\zeta = 0, \tag{27}$$

which is identical with the formula of ordinary analysis.[12]

I have gone at length into this very simple point in order to illustrate the fact, which I think is a general one, that the modern geometry is not only tending to

[12]The letters ξ, η, ζ, ω, here denote the distances of the plane Π from the points A, B, C, D, divided by six times the volume of the tetrahedron, A, B, C, D. The letters x, y, z, w, denote the tetrahedral coordinates as above.

results which are appropriately expressed in multiple algebra, but that it is actually striving to clothe itself in forms which are remarkably similar to the notations of multiple algebra, only less simple and general and far less amenable to analytical treatment, and therefore, that a certain logical necessity calls for throwing off the yoke under which analytical geometry has so long labored. And lest this should seem to be the utterance of an uninformed enthusiasm, or the echoing of the possibly exaggerated claims of the devotees of a particular branch of mathematical study, I will quote a sentence from Clebsch and one from Clifford, relating to the past and to the future of multiple algebra. The former is his eulogy on Plücker,[13] in 1871, speaking of recent advances in geometry, says that "in a certain sense the coordinates of a straight line, and in general a great part of the fundamental conceptions.of the newer algebra, are contained in the *Ausdehnungslehre* of 1844," and Clifford[14] in the last year of his life, speaking of the *Ausdehnungslehre*, with which he had but recently become acquainted, expresses "his profound admiration of that extraordinary work, and his conviction that its principles will exercise a vast influence upon the future of mathematical science."

Another subject in which we find a tendency toward the forms and methods of multiple algebra, is the calculus of operations. Our ordinary analysis introduces operators, and the successive operations A and B may be equivalent to the operation C. To express this in an equation we may write

$$BA(x) = C(x),$$

where x is any quantity or function. We may also have occasion to write

$$A(x) + B(x) = D(x), \text{ or } (A+B)(x) = D(x).$$

But it is almost impossible to resist the tendency to express these relations in the form

$$BA = C,$$

$$A + B = D,$$

in which the operators appear in a sense as quantities, i.e., as subjects of functional operation. Now since these operators are often of such nature that they cannot be perfectly specified by a single numerical quantity, when we treat them as quantities they must be regarded as multiple quantities. In this way certain formulæwhich essentially belong to multiple algebra get a precarious footing where they are only allowed because they are regarded as abridged notations for equations in ordinary

[13]*Gott. Abhandl.*, vol. xvi, p. 28.
[14]*Amer. Journ. Math.*, vol. i, p. 350.

algebra. Yet the logical development of such notations would lead a good way in multiple algebra, and doubtless many investigators have entered the field from this side.

One might also notice, to show how the ordinary algebra is becoming saturated with the notions and notations which seem destined to turn it into a multiple algebra, the notation so common in the higher algebra

$$(a,b,c)(x,y,z)$$

for

$$ax+by+cz.$$

This is evidently the same as Grassmann's internal product of the multiple quantities (a,b,c) and (x,y,z), or, in the language of quaternions, the scalar part, taken negatively, of the product of the vectors of which a,b,c and x,y,z are the components. A similar correspondence with Grassmann's methods might, I think, be shown in such notations as, for example,

$$(a,b,c,d)(x,y)^3.$$

The free admission of such notations is doubtless due to the fact that they are regarded simply as abridged notations.

The author of the celebrated "Memoir on the Theory of Matrices" goes much farther than this in his use of the forms of multiple algebra. Thus he writes explicitly one equation to stand for several, without the use of any of the analytical artifices which have been mentioned. This work has indeed, as we have seen, been characterized as marking the commencement of multiple algebra,—a view to which we can only take exception as not doing justice to earlier writers.

But the significance of this memoir with regard to the point which I am now considering is that it shows that the chasm so marked in the second quarter of this century is destined to be closed up. Notions and notations for which a Cayley is sponsor will not be excluded from good society among mathematicians. And if we admit as suitable the notations used in this memoir (where it is noticeable that the author rather avoids multiple algebra, and only uses it very sparingly), we shall logically be brought to use a great deal more. For example, if it is a good thing to write in our equations a single letter to represent a matrix of n^2 numerical quantities, why not use a single letter to represent the n quantities operated upon, as Grassmann and Hamilton have done? Logical consistency seems to demand it. And if we may use the sign $\times$ to denote an operation by which two sets of quantities are combined to form a third set, as is the case in this memoir, why not use other signs to denote other functional operations of which the result is a multiple quantity? If it be conceded that this is the proper method to follow where simplicity of conception, or brevity of expression, or ease of transformation is served thereby, our algebra will become in large part a multiple algebra.

We have considered the subject a good while from the outside; we have glanced at the principal events in the history of multiple algebra; we have seen how the course of modern thought seems to demand its aid, how it is actually leaning toward it, and beginning to adopt its methods. It may be worth while to direct our attention more critically to multiple algebra itself, and inquire into its essential character and its most important principles.

I do not know that anything useful or interesting, which relates to multiple quantity, and can be symbolically expressed, falls outside of the domain of multiple algebra. But if it is asked, what notions are to be regarded as fundamental, we must answer, here as elsewhere, those which are most simple and fruitful. Unquestionably, no relations are more so than those which are known by the names of addition and multiplication.

Perhaps I should here notice the essentially different manner in which the multiplication of multiple quantities has been viewed by different writers. Some, as Hamilton, or De Morgan, or Peirce speak of the product of two multiple quantities, as if only one product could exist, at least in the same algebra. Others, as Grassmann, speak of various kinds of products for the same multiple quantities. Thus Hamilton seems for many years to have agitated the question, what he should regard as the product of each pair of a set of triplets, or in the geometrical application of the subject, what he should regard as the product of each pair of a system of perpendicular directed lines.[15] Grassmann asks, What products, i.e., what distributive functions of the multiple quantities, are most important?

It may be that in some cases the fact that only one kind of product is known in ordinary algebra has led those to whom the problem presented itself in the form of finding a new algebra to adopt this characteristic derived from the old. Perhaps the reason lies deeper in a distinction like that in arithmetic between concrete and abstract numbers or quantities. The multiple quantities corresponding to concrete quantities such as ten apples or three miles are evidently such combinations as ten apples + seven oranges, three miles northward + five miles eastward, or six miles in a direction fifty degrees east of north. Such are the fundamental multiple quantities from Grassmann's point of view. But if we ask what it is in multiple algebra which corresponds to an abstract number like twelve, which is essentially an operator, which changes one mile into twelve miles, and $1,000 into 12,000$, the

[15] *Phil. Mag.* (3), vol. xxv, p. 490; *North British Review*, vol. xlv, (1866), p. 57.

most general answer would evidently be: an operator which will work such changes as, for example, that of ten apples + seven oranges into fifty apples + 100 oranges, or that of one vector into another.

Now an operator has, of course, one characteristic relation, viz., its relation to the operand. This needs no especial definition, since it is contained in the definition of the operator. If the operation is distributive, it may not inappropriately be called multiplication, and the result is *par excellence* the product of the operator and operand. The sum of operators *qua* operators, is an operator which gives for the product the sum of the products given by the operators to be added. The product of two operators is an operator which is equivalent to the successive operations of the factors. This multiplication is necessarily associative, and its definition is not really different from that of the operators themselves. And here I may observe that Professor C. S. Peirce has shown that his father's associative algebras may be regarded as operational and matricular.[16]

Now the calculus of distributive operators is a subject of great extent and importance, but Grassmann's view is the more comprehensive, since it embraces the other with something besides. For every quantitative operator may be regarded as a quantity, i.e., as the subject of mathematical operation, but every quantity cannot be regarded as an operator; precisely as in grammar every verb may be taken as substantive, as in the infinitive, while every substantive does not give us a verb.

Grassmann's view seems also the most practical and convenient. For we often use many functions of the same pair of multiple quantities, which are distributive with respect to both, and we need some simple designation to indicate a property of such fundamental importance in the algebra of such functions, and no advantage appears in singling out a particular function to be alone called the product. Even in quaternions, where Hamilton speaks of only one product of two vectors (regarding it as a special case of the product of quaternions, i.e., of operators), he nevertheless comes to use the scalar part of this product and the vector part separately. Now the distributive law is satisfied by each of these, which therefore may conveniently be called products. In this sense we have three kinds of products of vectors in Hamilton's analysis.

Let us then adopt the more general view of multiplication, and call any function of two or more multiple quantities, which is distributive with respect to all, a product, with only this limitation, that when one of the factors is simply an ordinary algebraic quantity, its effect is to be taken in the ordinary sense.

It is to be observed that this definition of multiplication implies that we have an addition both of the kind of quantity to which the product belongs, and of the kinds

[16] *Amer. Journ. Math.*, vol. iv, p. 221.

to which the factors belong. Of course, these must be subject to the general formal laws of addition. I do not know that it is necessary for the purposes of a general discussion to stop to define these operations more particularly, either on their own account or to complete the definition of multiplication. Algebra, as a formal science, may rest on a purely formal foundation. To take our illustration again from mechanics, we may say that if a man is inventing a particular machine,—a sewing machine, a reaper,—nothing is more important than that he should have a precise idea of the operation which his machine is to perform, yet when he is treating the general principles of mechanics he may discuss the lever, or the form of the teeth of wheels which will transmit uniform motion, without inquiring the purpose to which the apparatus is to be applied; and in like manner that if we were forming a particular algebra,—a geometrical algebra, a mechanical algebra, an algebra for the theory of elimination and substitution, an algebra for the study of quantics,—we should commence by asking, What are the multiple quantities, or sets of quantities, which we have to consider? What are the additive relations between them? What are the multiplicative relations between them? etc., forming a perfectly defined and complete idea of these relations as we go along; but in the development of a general algebra no such definiteness of conception is requisite. Given only the purely formal law of the distributive character of multiplication,—this is sufficient for the foundation of a science. Nor will such a science be merely a pastime for an ingenious mind. It will serve a thousand purposes in the formation of particular algebras. Perhaps we shall find that in the most important cases the particular algebra is little more than an application or interpretation of the general.

Grassmann observes that any kind of multiplication of n-fold quantities is characterized by the relations which hold between the products of n independent units. In certain kinds of multiplication these characteristic relations will hold true of the products of any of the quantities.

Thus if the value of a product is independent of the order of the factors when these belong to the system of units, it will always be independent of the order of the factors. The kind of multiplication characterized by this relation and no other between the products is called by Grassmann *algebraic*, because its rules coincide with those of ordinary algebra. It is to be observed, however, that it gives rise to multiple quantities of higher orders. If n independent units are required to express the original quantities, $n(n+1)/2$ units will be required for the products of two factors, $n(n+1)(n+2)/(2.3)$ for the products of three factors, etc.

Again, if the value of a product of factors belonging to a system of units is multiplied by -1 when two factors change places, the same will be true of the product of any factors obtained by addition of the units. The kind of multiplication characterized by this relation and no other is called by Grassmann *external* or *com-*

binatorial. For our present purpose we may denote it by the sign $\times$. It gives rise to multiple quantities of higher orders, $n(n-1)/2$ units being required to express the products of two factors, $n(n-1)(n-2)/(2.3)$ units for products of three factors, etc. All products of more than n factors are zero. The products of n factors may be expressed by a single unit, viz., the product of the n original units taken in a specified order, which is generally set equal to 1. The products of $n-1$ factors are expressed in terms of n units, those of $n-2$ factors in terms of $n(n-1)/2$ units, etc. This kind of multiplication is associative, like the algebraic.

Grassmann observes, with respect to binary products, that these two kinds of multiplication are the only kinds characterized by laws which are the same for any factors as for particular units; except indeed that characterized by no special laws, and that for which all products are zero.[17] The last we may evidently reject as nugatory. That for which there are no special laws, i.e., in which no equations subsist between the products of a system of independent units, is also rejected by Grassmann, as not appearing to afford important applications. I shall, however, have occasion to speak of it, and shall call it the indeterminate product. In this kind of multiplication, n^2 units are required to express the products of two factors, and n^3 units for products of three factors, etc. It evidently may be regarded as associative.

Another very important kind of multiplication is that called by Grassmann *internal*. In the form in which I shall give it, which is less general than Grassmann's, it is in one respect the most simple of all, since its only result is a numerical quantity. It is essentially binary and characterized by laws of the form

$$i.i = 1, j.j = 1, k.k = 1, \text{etc.},$$

$$i.j = 0, j.i = 0, \text{etc.},$$

where i, j, k, etc., represent a system of independent units. I use the dot as significant of this kind of multiplication.

Grassmann derives this kind of multiplication from the combinatorial by the following process. He defines the complement (Ergänzung) of a unit as the combinatorial product of all the other units, taken with such a sign that the combinatorial product of the unit and its complement shall be positive. The combinatorial product of a unit and its complement is therefore unity, and that of a unit and the complement of any other unit is zero. The internal product of two units is the combinatorial product of the first and the complement of the second.

It is important to observe that any scalar product of two factors of the same kind of multiple quantities, which is positive when the factors are identical, may

[17] Crelle's *Journ. f. Math.*, vol. xlix, p. 138.

be regarded as an internal product, i.e., we may always find such a system of units, that the characteristic equations of the product will reduce to the above form. The nature of the subject may afford a definition of the product independent of any reference to a system of units. Such a definition will then have obvious advantages. An important case of this kind occurs in geometry in that product of two vectors which is obtained by multiplying the products of their lengths by the cosine of the angle which they include. This is an internal product in Grassmann's sense.

Let us now return to the indeterminate product, which I am inclined to regard as the most important of all, since we may derive from it the algebraic and the combinatorial. For this end we will prefix $\sum$ to an indeterminate product to denote the sum of all the terms obtained by taking the factors in every possible order. Then,

$$\sum \alpha|\beta|\gamma,$$

for instance, where the vertical line is used to denote the indeterminate product,[18] is a distributive function of α, β, and γ. It is evidently not affected by changing the order of the letters. It is, therefore, an algebraic product in the sense in which the term has been defined.

So, again, if we prefix $\sum \pm$ to an indeterminate product to denote the sum of all terms obtained by giving the factors every possible order, those terms being taken negatively which are obtained by an odd number of simple permutations,

$$\sum \pm \alpha|\beta|\gamma,$$

for instance, will be a distributive function of α, β, γ, which is multiplied by -1 when two of these letters change places. It will therefore be a combinatorial product.

It is a characteristic and very important property of an indeterminate product that every product of all its factors with any other quantities is also a product of the indeterminate product and the other quantities. We need not stop for a formal proof of this proposition, which indeed is an immediate consequence of the definitions of the terms.

These considerations bring us naturally to what Grassmann calls *regressive multiplication*, which I will first illustrate by a very simple example. If n, the degree of multiplicity of our original quantities, is 4, the combinatorial product of $\alpha \times \beta \times \gamma$ and $\delta \times \varepsilon$, viz.,

$$\alpha \times \beta \times \gamma \times \delta \times \varepsilon,$$

[18]This notation must not be confounded with Grassmann's use of the vertical line.

is necessarily zero, since the number of factors exceeds four. But if for $\delta \times \varepsilon$ we set its equivalent

$$\delta|\varepsilon - \varepsilon|\delta,$$

we may multiply the first factor in each of these indeterminate products combinatorially by $\alpha \times \beta \times \gamma$, and prefix the result, which is a numerical quantity, as coefficient to the second factor. This will give

$$(\alpha \times \beta \times \gamma \times \delta)\varepsilon - (\alpha \times \beta \times \gamma \times \varepsilon)\delta.$$

Now, the first term of this expression is a product of $\alpha \times \beta \times \gamma, \delta$, and ε, and therefore, by the principle just stated, a product of $\alpha \times \beta \times \gamma$ and $\delta|\varepsilon$. The second term is a similar product of $\alpha \times \beta \times \gamma$ and $\varepsilon|\delta$. Therefore the whole expression is a product of $\alpha \times \beta \times \gamma$ and $\delta|\varepsilon - \varepsilon|\delta$, that is, of $\alpha \times \beta \times \gamma$ and $\delta \times \varepsilon$. That is, except in sign, what Grassmann calls the *regressive product* of $\alpha \times \beta \times \gamma$ and $\delta \times \varepsilon$.

To generalize this process, we first observe that an expression of the form

$$\sum \pm \alpha \times \beta|\gamma \times \delta,$$

in which each term is an indeterminate product of two combinatorial products, and in which $\sum \pm$ denotes the sum of all terms obtained by putting every different pair of the letters before the dividing line, the negative sign being used for any terms which may be obtained by an odd number of simple permutations of the letters,—in other words, the expression

$$\alpha \times \beta|\gamma \times \delta - \alpha \times \gamma|\beta \times \delta - \alpha \times \delta|\gamma \times \beta + \beta \times \gamma|\alpha \times \delta$$

$$-\beta \times \delta|\alpha \times \gamma + \gamma \times \delta|\alpha \times \beta,$$

is a distributive function of α, β, γ, and δ, which is multiplied by -1 when two of these letters change places, and may, therefore, be regarded as equivalent to the combinatorial product $\alpha \times \beta \times \gamma \times \delta$. Now, if $n = 5$, the combinatorial product of

$$\rho \times \sigma \times \tau \text{ and } \alpha \times \beta \times \gamma \times \delta$$

is zero. But if we multiply the first member of each of the above indeterminate products by $\rho \times \sigma \times \tau$, and prefix the result as coefficient to the second member, we obtain

$$(\rho \times \sigma \times \tau \times \alpha \times \beta)\gamma \times \delta - (\rho \times \sigma \times \tau \times \alpha \times \gamma)\beta \times \delta + \text{etc.},$$

which is what Grassmann calls the regressive product of $\rho \times \sigma \times \tau$ and $\alpha \times \beta \times \gamma \times \delta$. It is easy to see that the principle may be extended so as to give a regressive product in any case in which the total number of factors of two combinatorial products is greater than n. Also, that we might form a regressive product by treating the first of the given combinatorials as we have treated the second. It may easily be shown that this would give the same result, except in some cases with a difference of sign. To avoid this inconvenience, we may make the rule, that whenever in the substitution of a sum of indeterminate products for a combinatorial, both factors of the indeterminate products are of odd degree, we change the sign of the whole expression. With this understanding, the results which we obtain will be identical with Grassmann's regressive product. The propriety of the name consists in the fact that the product is of less degree than either of the factors. For the contrary reason, the ordinary external or combinatorial multiplication is sometimes called by Grassmann *progressive*.

Regressive multiplication is associative and exhibits a very remarkable analogy with the progressive. This analogy I have not time here to develop, but will only remark that in this analogy lies in its most general form that celebrated *principle of duality*, which appears in various forms in geometry and certain branches of analysis.

To fix our ideas, I may observe that in geometry the progressive multiplication of points gives successively lines, planes and volumes; the regressive multiplication of planes gives successively lines, points and scalar quantities.

The indeterminate product affords a natural key to the subject of matrices. In fact, a sum of indeterminate products of the second degree represents n^2 scalars, which constitute an ordinary or quadratic matrix; a sum of indeterminate products of the third degree represents n^3 scalars, which constitute a cubic matrix, etc. I shall confine myself to the simplest and most important case, that of quadratic matrices.

An expression of the form

$$\alpha(\lambda.\rho)$$

being a product of α, λ, and ρ. may be regarded as a product of $\alpha|\lambda$ and ρ, by a principle already stated. Now if Φ denotes a sum of indeterminate products, of second degree, say $\alpha|\lambda+\beta|\mu+$ etc., we may write

$$\Phi.\rho$$

for

$$\alpha(\lambda.\rho)+\beta(\mu.\rho)+\text{etc.}$$

This is like ρ, a quantity of the first degree, and it is a homogeneous linear function of ρ. It is easy to see that the most general form of such a function may be expressed in this way. An equation like

$$\sigma = \Phi.\rho$$

represents n equations in ordinary algebra, in which n variables are expressed as linear functions of n others by means of n^2 coefficients.

The internal product of two indeterminate products may be defined by the equation

$$(\alpha|\beta).(\gamma|\delta) = (\beta.\gamma)\alpha|\delta.$$

This defines the internal product of matrices, as

$$\Psi.\Phi.$$

This product evidently gives a matrix, the operation of which is equivalent to the successive operations of Φ and Ψ; i.e.,

$$(\Psi.\Phi).\rho = \Psi.(\Phi.\rho)$$

We may express this a little more generally by saying that internal multiplication is associative when performed on a series of matrices, or on such a series terminated by a quantity of the first degree.

Another kind of multiplication of binary indeterminate products is that in which the preceding factors are multiplied combinatorially, and also the following. It may be defined by the equation

$$(\alpha|\lambda)_\times^\times(\beta|\mu)_\times^\times(\gamma|\nu) = \alpha\times\beta\times\gamma|\lambda\times\mu\times\nu.$$

This defines a multiplication of matrices denoted by the same symbol, as

$$\Phi_\times^\times\Psi_\times^\times\Omega, \Phi_\times^\times\Psi_\times^\times\Omega_\times^\times\Theta.$$

This multiplication, which is associative and commutative, is of great importance in the theory of determinants. In fact,

$$\frac{1}{\underline{|n}}\Phi_\times^{\times n}$$

is the determinant of the matrix Φ. A lower power, as the m'th, with the divisor $n(n-1)\ldots(n-m+1)$ would express as multiple quantity all the subdeterminants of order m.[19]

[19]Quadratic matrices may also be represented by a sum of indeterminate products of a quantity of the first degree with a combinatorial product of $(n-1)$st degree, as, for example, when $n = 4$, by a sum of products of the form

$$\alpha|\beta\times\gamma\times\delta.$$

The theory of such matrices is almost identical with that of those of the other form, except that the external multiplication takes the place of the internal, in the multiplication of the matrices with each other and with quantities of the first degree.

It is evident that by the combination of the operations of indeterminate, algebraic, and combinatorial multiplication we obtain multiple quantities of a more complicated nature than by the use of only one of these kinds of multiplication. The indeterminate product of combinatorial products we have already mentioned. The combinatorial product of algebraic products, and the indeterminate product of algebraic products, are also of great importance, especially in the theory of quantics. These three multiplications, with the internal, especially in connection with the general property of the indeterminate product given above, and the derivation of the algebraic and combinatorial products from the indeterminate, which affords a generalization of that property, give rise to a great wealth of multiplicative relations between these multiple quantities. I say "*wealth* of multiplicative relations" designedly, for there is hardly any kind of relations between things which are the objects of mathematical study, which add so much to the resources of the student as those which we call multiplicative, except perhaps the simpler class which we call additive, and which are presupposed in the multiplicative. This is a truth quite independent of our using any of the notations of multiple algebra, although a suitable notation for such relations will of course increase their value.

Perhaps, before closing, I ought to say a few words on the applications of multiple algebra.

First of all, geometry, and the geometrical sciences which treat of things having position in space, kinematics, mechanics, astronomy physics, crystallography, seem to demand a method of this kind, for position in space is essentially a multiple quantity and can only be represented by simple quantities in an arbitrary and cumbersome manner. For this reason, and because our spatial intuitions are more developed than those of any other class of mathematical relations, these subjects are especially adapted to introduce the student to the methods of multiple algebra. Here, Nature herself takes us by the hand and leads us along by easy steps, as a mother teaches her child to walk. In the contemplation of such subjects, Möbius, Hamilton, and Grassmann formed their algebras, although the philosophical mind of the last was not satisfied until he had produced a system unfettered by any spatial relations. It is probably in connection with some of these subjects that the notions of multiple algebra are most widely disseminated.

Maxwell's *Treatise on Electricity and Magnetism* has done so much to familiarize students of physics with quaternion notations, that it seems impossible that this subject should ever again be entirely divorced from the methods of multiple algebra.

I wish that I could say as much of astronomy. It is, I think, to be regretted, that the oldest of the scientific applications of mathematics, the most dignified, the most conservative, should keep so far aloof from the youngest of mathematical methods;

and standing as I do to-day, by some chance, among astronomers, although not of the guild, I cannot but endeavor to improve the opportunity by expressing my conviction of the advantages which astronomers might gain by employing some of the methods of multiple algebra. A very few of the fundamental notions of a vector analysis, the addition of vectors and what quaternionists would call the scalar part and the vector part of the product of two vectors (which may be defined without the notion of the quaternion),—these three notions with some four fundamental properties relating to them are sufficient to reduce enormously the labor of mastering such subjects as the elementary theory of orbits, the determination of an orbit from three observations, the differential equations which are used in determining the best orbit from an indefinite number of observations by the method of least squares, or those which give the perturbations when the elements are treated as variable. In all these subjects the analytical work is greatly simplified, and it is far easier to find the best form for numerical calculation than by the use of the ordinary analysis.

I may here remark that in its geometrical applications multiple algebra will naturally take one of two principal forms, according as vectors or points are taken as elementary quantities, i.e., according as something having magnitude and direction, or something having magnitude and position at a point, is the fundamental conception. These forms of multiple algebra may be distinguished as *vector analysis* and *point analysis*. The former we may call a triple, the latter a quadruple algebra, if we determine the degree of the algebra from the degree of multiplicity of the fundamental conception. The former is included in the latter, since the subtraction of points gives us vectors, and in this way Grassmann's vector analysis is included in his point analysis. Hamilton's system, in which the vector is the fundamental idea, is nevertheless made a quadruple algebra by the addition of ordinary numerical quantities. For practical purposes we may regard Hamilton's system as equivalent to Grassmann's algebra of vectors. Such practical equivalence is of course consistent with great differences of notation, and of the point of view from which the subject is regarded.

Perhaps I should add a word in regard to the nature of the problems which require a vector analysis, or the more general form of Grassmann's point analysis. The distinction of the problems is very marked, and corresponds precisely to the distinction familiar to all analysts between problems which are suitable for Cartesian coordinates, and those which are suitable for the use of tetrahedral, or, in plane geometry, triangular coordinates. Thus, in mechanics, kinematics, astronomy, physics, or crystallography, Grassmann's point analysis will rarely be wanted. One might teach these subjects for years by a vector analysis, and never perhaps feel the need of any of the notions or notations which are peculiar to the point analysis, precisely as in ordinary algebra one might use the Cartesian coordinates in teaching these subjects, without any occasion for the use of tetrahedral coordinates. I think of one exception, which, however, confirms the rule. The very important theory of forces acting on a rigid body is much better treated by point analysis than by vector analysis, exactly as in ordinary algebra it is much better treated by tetrahedral coordinates than by Cartesian,—I mean for the purpose of the elegant development of general propositions. A sufficient theory for the purposes of numerical calculations can easily enough be given by any method, and the most familiar to the student is for such practical purposes of course the best. On the other hand, the projective properties of bodies, the relations of collinearity, and similar subjects, seem to demand the point analysis for their adequate treatment.

If I have said that the algebra of vectors is contained in the algebra of points, it does not follow that in a certain sense the algebra of points is not deducible from the algebra of vectors. In mathematics, a part often contains the whole. If we represent points by vectors drawn from a common origin, and then develop those relations between such vectors representing points, which are independent of the position of the origin,—by this simple process we may obtain a large part, possibly all, of an algebra of points. In this way the vector analysis may be made to serve very conveniently for many of those subjects which I have mentioned as suitable for point analysis. The vector analysis, thus enlarged, is hardly to be distinguished from a point analysis, but the treatment of the subject in this way has somewhat of a makeshift character, as distinguished from the unity and simplicity of the subject when developed directly from the idea of something situated at a point.

Of those subjects which have no relations to space, the elementary theory of eliminations and substitutions, including the theory of matrices and determinants, seems to afford the most simple application of multiple algebra. I have already indicated what seems to me the appropriate foundation for the theory of matrices. The method is essentially that which Grassmann has sketched in his first *Ausdehnungslehre* under the name of the *open product* and has developed at length in the second.

In the theory of quantics Grassmann's algebraic product finds an application, the quantic appearing as a sum of algebraic products in Grassmann's sense of the term. As it has been stated that these products are subject to the same laws as the ordinary products of algebra, it may seem that we have here a distinction without an important difference. If the quantics were to be subject to no farther multiplications, except the algebraic in Grassmann's sense, such an objection would be valid. But quantics regarded as sums of algebraic products, in Grassmann's sense, are multiple quantities and subject to a great variety of other multiplications than the algebraic, by which they were formed. Of these the most important are doubtless the combinatorial, the internal, and the indeterminate. The combinatorial and the internal may be applied, not only to the quantic as a whole or to the algebraic products of which it consists, but also to the individual factors in each term, in accordance with the general principle which has been stated with respect to the indeterminate product and which will apply also to the algebraic, since the algebraic may be regarded as a sum of indeterminate products.

In the differential and integral calculus it is often advantageous to regard as multiple quantities various sets of variables, especially the independent variables, or those which may be taken as such. It is often convenient to represent in the form of a single differential coefficient, as

$$\frac{d\tau}{d\rho},$$

a block or matrix of ordinary differential coefficients. In this expression, ρ may be a multiple quantity representing say n independent variables, and τ another representing perhaps the same number of dependent variables. Then $d\rho$ represents the n differentials of the former, and $d\tau$ the n differentials of the latter. The whole expression represents an operator which turns $d\rho$ into $d\tau$, so that we may write identically

$$d\tau = \frac{d\tau}{d\rho} d\rho.$$

Here we see a matrix of n^2 differential coefficients represented by a quotient. This conception is due to Grassmann, as well as the representation of the matrix by a sum of products, which we have already considered. It is to be observed that these multiple differential coefficients are subject to algebraic laws very similar to those which relate to ordinary differential coefficients when there is a single independent variable, e.g.,

$$\frac{d\sigma}{d\tau}\frac{d\tau}{d\rho} = \frac{d\sigma}{d\rho},$$

$$\frac{d\rho}{d\tau}\frac{d\tau}{d\rho} = 1.$$

In the integral calculus, the transformation of multiple integrals by change of variables is made very simple and clear by the methods of multiple algebra.

In the geometrical applications of the calculus, there is a certain class of theorems, of which Green's and Poisson's are the most notable examples, which seem to have been first noticed in connection with certain physical theories, especially those of electricity and magnetism, and which have only recently begun to find their way into treatises on the calculus. These not only find simplicity of expression and demonstration in the infinitesimal calculus of multiple quantities, but also their natural position, which they hardly seem to find in the ordinary treatises.

But I do not so much desire to call your attention to the diversity of the applications of multiple algebra, as to the simplicity and unity of its principles. The student of multiple algebra suddenly finds himself freed from various restrictions to which he has been accustomed. To many, doubtless, this liberty seems like an invitation to license. Here is a boundless field in which caprice may riot. It is not strange if some look with distrust for the result of such an experiment. But the farther we advance, the more evident it becomes that this too is a realm subject to law. The more we study the subject, the more we find all that is most useful and beautiful attaching itself to a few central principles. We begin by studying *multiple algebras*; we end, I think, by studying MULTIPLE ALGEBRA.