ELEMENTS OF VECTOR ANALYSIS

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(The fundamental principles of the following analysis are such as are familiar under a slightly different form to students of quaternions. The manner in which the subject is developed is somewhat different from that followed in treatises on quaternions, since the object of the writer does not require any use of the conception of the quaternion, being simply to give a suitable notation for those relations between vectors, or between vectors and scalars, which seem most important, and which lend themselves most readily to analytical transformations, and to explain some of these transformations. As a precedent for such a departure from quaternionic usage, Clifford's *Kinematic* may be cited. In this connection, the name of Grassmann may also be mentioned, to whose system the following method attaches itself in some respects more closely than to that of Hamilton.)

Chapter 1

CONCERNING THE ALGEBRA OF VECTORS.

Fundamental Notions.

1. *Definition*.—If anything has magnitude and direction, its magnitude and direction taken together constitute what is called a vector.

The numerical description of a vector requires three numbers, but nothing prevents us from using a single letter for its symbolical designation. An algebra or analytical method in which a single letter or other expression is used to specify a vector may be called a *vector algebra* or *vector analysis*.

Def.—As distinguished from vectors the real (positive or negative) quantities of ordinary algebra are called *scalars*.[1]

As it is convenient that the form of the letter should indicate whether a vector or a scalar is denoted, we shall use the small Greek letters to denote vectors, and the small English letters to denote scalars. (The three letters, i, j, k, will make an exception, to be mentioned more particularly hereafter. Moreover, π will be used in its usual scalar sense, to denote the ratio of the circumference of a circle to its diameter.)

2. *Def.*—Vectors are said to be *equal* when they are the same both in direction and in magnitude. This equality is denoted by the ordinary sign, as $\alpha = \beta$. The reader will observe that this *vector equation* is the equivalent of three scalar equations.

A vector is said to be equal to zero, when its magnitude is zero. Such vectors may be set equal to one another, irrespectively of any considerations relating to direction.

3. Perhaps the most simple example of a vector is afforded by a directed straight line, as the line drawn from A to B. We may use the notation $\overline{AB}$ to denote this line as a vector, i.e., to denote its length and direction without regard to its position in other respects. The points A and B may be distinguished as the *origin* and the *terminus* of the vector. Since any magnitude may be represented by a length, any vector may be

[1]The imaginaries of ordinary algebra may be called *biscalars*, and that which corresponds to them in the theory of vectors, *bivectors*. But we shall have no occasion to consider either of these. [See, however, footnote to "Note on Bivector Analysis", after No. 189.]

represented by a directed line; end it will often be convenient to use language relating to vectors, which refers to them as thus represented.

Reversal of Direction, Scalar Multiplication and Division.

4. The negative sign ($-$) reverses the direction of a vector. (Sometimes the sign $+$ may be used to call attention to the fact that the vector has not the negative sign.)

Def.—A vector is said to be *multiplied* or *divided by a scalar* when its magnitude is multiplied or divided by the numerical value of the scalar and its direction is either unchanged or reversed according as the scalar is positive or negative. These operations are represented by the same methods as multiplication and division in algebra, and are to be regarded as substantially identical with them. The terms *scalar multiplication* and *scalar division* are used to denote multiplication and division by scalars, whether the quantity multiplied or divided is a scalar or a vector.

5. *Def.*—A unit vector is a vector of which the magnitude is unity.

Any vector may be regarded as the product of a positive scalar the magnitude of the vector) and a unit vector.

The notation a_0 may be used to denote the magnitude of the vector a.

Addition and Subtraction of Vectors.

6. *Def.*—The *sum* of the vectors α, β, etc. (written $\alpha + \beta + \mathrm{etc}.$) is the vector found by the following process. Assuming any point A, we determine successively the points B, C, etc., so that $\overline{AB} = \alpha$, $\overline{BC} = \beta$, etc. The vector drawn from A to the last point thus determined is the sum required. This is sometimes called the *geometrical* sum, to distinguish it from an *algebraic* sum or an *arithmetical* sum. It is also called the resultant, and α, β, etc. are called the components. When the vectors to be added are all parallel to the same straight line, geometrical addition reduces to algebraic; when they have all the same direction, geometrical addition like algebraic reduces to arithmetical.

It may easily be shown that the value of a sum is not affected by changing the order of two consecutive terms, and therefore that it is not affected by any change in the order of the terms. Again, it is evident from the definition that the value of a sum is not altered by uniting any of its terms in brackets, as $\alpha + [\beta + \gamma] + \mathrm{etc}.$, which is in effect to substitute the sum of the terms enclosed for the terms themselves among the vectors to be added. In other words, the commutative and associative principles of arithmetical and algebraic addition hold true of geometrical addition.

7. *Def.*—A vector is said to be subtracted when it is added after reversal of direction. This is indicated by the use of the sign $-$ instead of $+$.

8. It is easily shown that the distributive principle of arithmetical and algebraic multiplication applies to the multiplication of sums of vectors by scalars or sums of

scalars, i.e.,

$$\begin{aligned}(m+n+\text{etc}.)[\alpha+\beta+\text{etc}.] = m\alpha + n\beta + \text{etc}.\\ +m\beta + n\beta + \text{etc}.\\ +\text{etc}.\end{aligned}$$

9. *Vector Equations*.—If we have equations between sums and differences of vectors, we may transpose terms in them, multiply or divide by any scalar, and add or subtract the equations, precisely as in the case of the equations of ordinary algebra. Hence, if we have several such equations containing known and unknown vectors, the processes of elimination and reduction by which the unknown vectors may be expressed in terms of the known are precisely the same, and subject to the same limitations, as if the letters representing vectors represented scalars. This will be evident if we consider that in the multiplications incident to elimination in the supposed scalar equations the multipliers are the coefficients of the unknown quantities, or functions of these coefficients, and that such multiplications may be applied to the vector equations, since the coefficients are scalars.

10. *Linear relation of four vectors, Coordinates*.—If α, β, and γ are any given vectors not parallel to the same plane, any other vector ρ may be expressed in the form

$$\rho = a\alpha + b\beta + c\gamma.$$

If α, β, and γ are unit vectors, a, b, and c are the ordinary scalar components of ρ parallel to α, β, and γ. If $\rho = \overline{OP}$, (α, β, γ being unit vectors), a, b, and c are the cartesian coordinates of the point P referred to axes through O parallel to α, β, and γ. When the values of these scalars are given, ρ is said to be given in terms of α, β, and γ. It is generally in this way that the value of a vector is specified, viz., in terms of three known vectors. For such purposes of reference, a system of three mutually perpendicular vectors has certain evident advantages.

11. *Normal systems of unit vectors*.—The letters i, j, k are appropriated to the designation of a *normal system of unit vectors*, i.e., three unit vectors, each of which is at right angles to the other two and determined in direction by them in a perfectly definite manner. We shall always suppose that k is on the side of the $i-j$ plane on which a rotation from i to j (through one right angle) appears counter-clockwise. In other words, the directions of i, j, and k are to be so determined that if they be turned (remaining rigidly connected with each other) so that i points to the east, and j to the north, k will point upward. When rectangular axes of X, Y, and Z are employed, their directions will be conformed to a similar condition, and i, j, k (when the contrary is not stated) will be supposed parallel to these axes respectively. We may have occasion to use more than one such system of unit vectors, just as we may use more than one system of coordinate axes. In such cases, the different systems may be distinguished by accents or otherwise.

12. *Numerical computation of a geometrical sum*.—If

$$\begin{aligned}&\rho = a\alpha + b\beta + c\gamma,\\ &\sigma = a'\alpha + b'\beta + c'\gamma,\\ &\text{etc}.,\end{aligned}$$

then

$$\rho + \sigma + \text{etc.} = (a + a' + \text{etc.})\alpha + (b + b' + \text{etc.})\beta + (c + c' + \text{etc.})\gamma,$$

i.e., the coefficients by which a geometrical sum is expressed in terms of three vectors are the sums of the coefficients by which the separate terms of the geometrical sum are expressed in terms of the same three vectors.

Direct and Skew Products of Vectors

13. *Def.*—The *direct product* of α and β (written $\alpha.\beta$) is the scalar quantity obtained by multiplying the product of their magnitudes by the cosine of the angle made by their directions.

14. *Def.*—The *skew product* of α and β (written $\alpha \times \beta$) is a vector function of α and β. Its magnitude is obtained by multiplying the product of the magnitudes of α and β by the sine of the angle made by their directions. Its direction is at right angles to α and β, and on that side of the plane containing α and β (supposed drawn from a common origin) on which a rotation from α to β through an arc of less than $180°$ appears counter-clockwise.

The direction of $\alpha \times \beta$ may also be defined as that in which an ordinary screw advances as it turns so as to carry α toward β.

Again, if α be directed toward the east, and β lie in the same horizontal plane and on the north side of α, $\alpha \times \beta$ will be directed upward.

15. It is evident from the preceding definitions that

$$\alpha.\beta = \beta.\alpha, \qquad \text{and} \qquad \alpha \times \beta = -\beta \times \alpha.$$

16. Moreover,

$$[n\alpha].\beta = \alpha.[n\beta] = n[\alpha.\beta],$$

and

$$[n\alpha] \times \beta = \alpha \times [n\beta] = n[\alpha \times \beta].$$

The brackets may therefore be omitted in such expressions.

17. From the definitions of No. 11 it appears that

$$i.i = j.j = k.k = 1,$$

$$i.j = j.i = i.k = k.i = j.k = k.j = 0,$$

$$\begin{aligned} i \times i &= 0, & j \times j &= 0, & k \times k &= 0, \\ i \times j &= k, & j \times k &= i, & k \times i &= j, \\ j \times i &= -k, & k \times j &= -i, & i \times k &= -j. \end{aligned}$$

18. If we resolve β into two components β' and β'', of which the first is parallel and the second perpendicular to α, we shall have

$$\alpha.\beta = \alpha.\beta' \qquad \text{and} \qquad \alpha \times \beta = \alpha \times \beta''.$$

19.

$$\alpha.[\beta+\gamma] = \alpha.\beta + \alpha.\gamma \qquad \text{and} \qquad \alpha \times [\beta+\gamma] = \alpha \times \beta + \alpha \times \gamma.$$

To prove this, let $\sigma = \beta + \gamma$, and resolve each of the vectors β, γ, σ into two components, one parallel and the other perpendicular to α. Let these be β', β'', γ', γ'', σ', σ''. Then the equations to be proved will reduce by the last section to

$$\alpha.\sigma' = \alpha.\beta' + \alpha.\gamma' \qquad \text{and} \qquad \alpha \times \sigma'' = \alpha \times \beta'' + \alpha \times \gamma''.$$

Now since $\sigma = \beta + \gamma$ we may form a triangle in space, the sides of which shall be β, γ, and σ. Projecting this on a plane perpendicular to α, we obtain a triangle having the sides β'', γ'', and σ'', which affords the relation $\sigma'' = \beta'' + \gamma''$. If we pass planes perpendicular to α through the vertices of the first triangle, they will give on a line parallel to α segments equal to β', γ', σ'. Thus we obtain the relation $\sigma' = \beta' + \gamma'$. Therefore $\alpha.\sigma' = \alpha.\beta' + \alpha.\gamma'$, since all the cosines involved in these products are equal to unity. Moreover, if α is a unit vector, we shall evidently have $\alpha \times \sigma'' = \alpha \times \beta'' + \alpha \times \gamma''$, since the effect of the skew multiplication by α upon vectors in a plane perpendicular to α is simply to rotate them all $90°$ in that plane. But any case may be reduced to this by dividing both sides of the equation to be proved by the magnitude of α. The propositions are therefore proved.

20. Hence,

$$[\alpha+\beta].\gamma = \alpha.\gamma + \beta.\gamma, \qquad [\alpha+\beta] \times \gamma = \alpha \times \gamma + \beta \times \gamma,$$

$$[\alpha+\beta].[\gamma+\delta] = \alpha.\gamma + \alpha.\delta + \beta.\gamma + \beta.\delta,$$

$$[\alpha+\beta] \times [\gamma+\delta] = \alpha \times \gamma + \alpha \times \delta + \beta \times \gamma + \beta \times \delta;$$

and, in general, direct and skew products of sums of vectors may be expanded precisely as the products of sums in algebra, except that in skew products the order of the factors must not be changed without compensation in the sign of the term. If any of the terms in the factors have negative signs, the signs of the expanded product (when there is no change in the order of the factors) will be determined by the same rules as in algebra. It is on account of this analogy with algebraic products that these functions of vectors are called *products* and that other terms relating to multiplication are applied to them.

21. *Numerical calculation of direct and skew products*.—The properties demonstrated in the last two paragraphs (which may be briefly expressed by saying that the operations of direct and skew multiplication are distributive) afford the rule for the numerical calculation of a direct product, or of the components of a skew product, when the rectangular components of the factors are given numerically. In fact, if

$$\alpha = xi + yj + zk, \qquad \text{and} \qquad \beta = x'i + y'j + z'k;$$

$$\alpha.\beta = xx' + yy' + zz',$$

and
$$\alpha \times \beta = (yz' - zy')i + (zx' - xz')j + (xy' - yx')k.$$

22. *Representation of the area of a parallelogram by a skew product*.—It will be easily seen that $\alpha \times \beta$ represents in magnitude the area of the parallelogram of which

α and β (supposed drawn from a common origin) are the sides, and that it represents in direction the normal to the plane of the parallelogram on the side on which the rotation from α toward β appears counter-clockwise.

23. *Representation of the volume of a parallelopiped by a triple product.*—It will also be seen that $\alpha \times \beta . \gamma$[2] represents in numerical value the volume of the parallelopiped of which α, β, and γ (supposed drawn from a common origin) are the edges, and that the value of the expression is positive or negative according as γ lies on the side of the plane of α and β on which the rotation from α to β appears counterclockwise, or on the opposite side.

24. Hence,

$$\begin{aligned}\alpha \times \beta . \gamma = \beta \times \gamma . \alpha = \gamma . \alpha \times \beta = \alpha . \beta \times \gamma \\ = \beta . \gamma \times \alpha = -\beta \times \alpha . \gamma = -\gamma \times \beta . \alpha = -\alpha \times \gamma . \beta \\ = -\gamma . \beta \times \alpha = -\alpha . \gamma \times \beta = -\beta . \alpha \times \gamma .\end{aligned}$$

It will be observed that all the products of this type, which can be made with three given vectors, are the same in numerical value, and that any two such products are of the same or opposite character in respect to sign, according as the cyclic order of the letters is the same or different. The product vanishes when two of the vectors are parallel to the same line, or when the three are parallel to the same plane.

This kind of product may be called the scalar product of the three vectors. There are two other kinds of products of three vectors, both of which are vectors, viz., products of the type $(\alpha . \beta)\gamma$ or $\gamma(\alpha . \beta)$, and products of the type $\alpha \times [\beta \times \gamma]$ or $[\gamma \times \beta] \times \alpha$.

25.

$$i . j \times k = j . k \times i = k . i \times j = 1. \qquad i . k \times j = k . j \times i = j . i \times k = -1.$$

From these equations, which follow immediately from those of No. 17, the propositions of the last section might have been derived, viz., by substituting for α, β, and γ, respectively, expressions of the form $xi + yj + zk$, $x'i + y'j + z'k$, and $x''i + y''j + z''k$.[3] Such a method, which may be called *expansion in terms of* i, j, and k, will on many occasions afford very simple, although perhaps lengthy, demonstrations.

26. *Triple products containing only two different letters.*—The significance and the relations of $(\alpha . \alpha)\beta$, $(\alpha . \beta)\alpha$, and $\alpha \times [\alpha \times \beta]$ will be most evident, if we consider β as made up of two components, β' and β'', respectively parallel and perpendicular to α. Then

$$\beta = \beta' + \beta'',$$

$$(\alpha . \beta)\alpha = (\alpha . \beta')\alpha = (\alpha . \alpha)\beta',$$

$$\alpha \times [\alpha \times \beta] = \alpha \times [\alpha \times \beta''] = -(\alpha . \alpha)\beta''.$$

Hence, $$\alpha \times [\alpha \times \beta] = (\alpha . \beta)\alpha - (\alpha . \alpha)\beta.$$

[2] Since the sign $\times$ is only used between vectors, the skew multiplication in expressions of this kind is evidently to be performed first. In other words, the above expression must be interpreted as $[\alpha \times \beta] . \gamma$.

[3] The student who is familiar with the nature of determinants will not fail to observe that the triple product $\alpha . \beta \times \gamma$ is the determinant formed by the nine rectangular components of α, β, and γ, nor that the rectangular components of $\alpha \times \beta$ are determinants of the second order formed from the components of α and β (See the last equation of No. 21.)

27. *General relation of the vector products of three factors.*—In the triple product $\alpha \times [\beta \times \gamma]$ we may set

$$\alpha = l\beta + m\gamma + n\beta \times \gamma,$$

unless β and γ have the same direction. Then

$$\begin{aligned}\alpha \times [\beta \times \gamma] &= l\beta \times [\beta \times \gamma] + m\gamma \times [\beta \times \gamma] \\ &= l(\beta.\gamma)\beta - l(\beta.\beta)\gamma - m(\gamma.\beta)\gamma + m(\gamma.\gamma)\beta \\ &= (l\beta.\gamma + m\gamma.\gamma)\beta - (l\beta.\beta + m\gamma.\beta)\gamma.\end{aligned}$$

But $$l\beta.\gamma + m\gamma.\gamma = \alpha.\gamma, \quad \text{and} \quad l\beta.\beta + m\gamma.\beta = \alpha.\beta.$$

Therefore $$\alpha \times [\beta \times \gamma] = (\alpha.\gamma)\beta - (\alpha.\beta)\gamma,$$

which is evidently true, when β and γ have the same directions. It may also be written

$$[\gamma \times \beta] \times \alpha = \beta(\gamma.\alpha) - \gamma(\beta.\alpha).$$

28. This principle may be used in the transformation of more complex products. It will be observed that its application will always simultaneously eliminate, or introduce, two signs of skew multiplication.

The student will easily prove the following identical equations, which, although of considerable importance, are here given principally as exercises in the application of the preceding formulæ.

29.
$$\alpha \times [\beta \times \gamma] + \beta \times [\gamma \times \alpha] + \gamma \times [\alpha \times \beta] = 0.$$

30.
$$[\alpha \times \beta].[\gamma \times \delta] = (\alpha.\gamma)(\beta.\delta) - (\alpha.\delta)(\beta.\gamma),$$

31.
$$[\alpha \times \beta] \times [\gamma \times \delta] = (\alpha.\gamma \times \delta)\beta - (\beta.\gamma \times \delta)\alpha = (\alpha.\beta \times \delta)\gamma - (\alpha.\beta \times \gamma)\delta.$$

32.
$$\alpha \times [\beta \times [\gamma \times \delta]] = (\alpha.\gamma \times \delta)\beta - (\alpha.\beta)\gamma \times \delta = (\beta.\delta)\alpha \times \gamma - (\beta.\gamma)\alpha \times \delta.$$

33.
$$\begin{aligned}[\alpha \times \beta].[\gamma \times \delta] \times [\epsilon \times \xi] &= (\alpha.\beta \times \delta)(\gamma.\epsilon \times \xi) - (\alpha.\beta \times \gamma)(\delta.\epsilon \times \xi) \\ &= (\alpha.\beta \times \epsilon)(\xi.\gamma \times \delta) - (\alpha.\beta \times \xi)(\epsilon.\gamma \times \delta) \\ &= (\gamma.\delta \times \alpha)(\beta.\epsilon \times \xi) - (\gamma.\delta \times \beta)(\alpha.\epsilon \times \xi).\end{aligned}$$

34.
$$[\alpha \times \beta].[\beta \times \gamma] \times [\gamma \times \alpha] = (\alpha.\beta \times \gamma)^2.$$

35. The student will also easily convince himself that a product formed of any number of letters (representing vectors) combined in any possible way by scalar, direct, and skew multiplications may be reduced by the principles of Nos. 24 and 27 to a sum of products, each of which consists of scalar factors of the forms $\alpha.\beta$ and $\alpha.\beta\times\gamma$, with a single vector factor of the form α or $\alpha\times\beta$, when the original product is a vector.

36. *Elimination of scalars from vector equations.*—It has already been observed that the elimination of vectors from equations of the form

$$a\alpha + b\beta + c\gamma + d\delta + \text{ etc. } = 0$$

is performed by the same rule as the eliminations of ordinary algebra. (See No. 9.) But the elimination of scalars from such equations is at least formally different. Since a single vector equation is the equivalent of three scalar equations, we must be able to deduce from such an equation a scalar equation from which two of the scalars which appear in the original vector equation have been eliminated. We shall see how this may be done, if we consider the scalar equation

$$a\alpha.\lambda + b\beta.\lambda + c\gamma.\lambda + d\delta.\lambda + \text{ etc. } = 0,$$

which is derived from the above vector equation by direct multiplication by a vector λ. We may regard the original equation as the equivalent of the three scalar equations obtained by substituting for α, β, γ, δ, etc., their X-, Y-, and Z-components. The second equation would be derived from these by multiplying them respectively by the X-, Y-, and Z-components of λ and adding. Hence the second equation may be regarded as the most general form of a scalar equation of the first degree in a, b, c, d, etc., which can be derived from the original vector equation or its equivalent three scalar equations. If we wish to have two of the scalars, as b and c, disappear, we have only to choose for λ a vector perpendicular to β and γ. Such a vector is $\beta\times\gamma$. We thus obtain

$$a\alpha.\beta\times\gamma + d\delta.\beta\times\gamma + \text{etc.} = 0.$$

37. *Relations of four vectors.*—By this method of elimination we may find the values of the coefficients a, b, and c in the equation

$$\rho = a\alpha + b\beta + c\gamma, \tag{1}$$

by which any vector ρ is expressed in terms of three others. (See No. 10.) If we multiply *directly* by $\beta\times\gamma$, $\gamma\times\alpha$, and $\alpha\times\beta$, we obtain

$$\rho.\beta\times\gamma = a\alpha.\beta\times\gamma, \quad \rho.\gamma\times\alpha = b\beta.\gamma\times\alpha, \quad \rho.\alpha\times\beta = c\gamma.\alpha\times\beta; \tag{2}$$

whence

$$a = \frac{\rho.\beta\times\gamma}{\alpha.\beta\times\gamma}, \quad b = \frac{\rho.\gamma\times\alpha}{\alpha.\beta\times\gamma}, \quad c = \frac{\rho.\alpha\times\beta}{\alpha.\beta\times\gamma}. \tag{3}$$

By substitution of these values, we obtain the identical equation,

$$(\alpha.\beta\times\gamma)\rho = (\rho.\beta\times\gamma)\alpha + (\rho.\gamma\times\alpha)\beta + (\rho.\alpha\times\beta)\gamma. \tag{4}$$

(Compare No. 31.) If we wish the four vectors to appear symmetrically in the equation we may write

$$(\alpha.\beta \times \gamma)\rho - (\beta.\gamma \times \rho)\alpha + (\gamma.\rho \times \alpha)\beta - (\rho.\alpha \times \beta)\gamma = 0. \tag{5}$$

If we wish to express ρ as a sum of vectors having directions perpendicular to the planes of α and β, of β and γ, and of γ and α, we may write

$$\rho = e\beta \times \gamma + f\gamma \times \alpha + g\alpha \times \beta. \tag{6}$$

To obtain the values of e, f, g, we multiply *directly* by α, by β and by γ. This gives

$$e = \frac{\rho.\alpha}{\beta.\gamma \times \alpha}, \quad f = \frac{\rho.\beta}{\gamma.\alpha \times \beta}, \quad g = \frac{\rho.\gamma}{\alpha.\beta \times \gamma}. \tag{7}$$

Substituting these values we obtain the identical equation

$$(\alpha.\beta \times \gamma)\rho = (\rho.\alpha)\beta \times \gamma + (\rho.\beta)\gamma \times \alpha + (\rho.\gamma)\alpha \times \beta. \tag{8}$$

(Compare No. 32.)

38. *Reciprocal systems of vectors.*—The results of the preceding section may be more compactly expressed if we use the abbreviations

$$\alpha' = \frac{\beta \times \gamma}{\alpha.\beta \times \gamma}, \quad \beta' = \frac{\gamma \times \alpha}{\beta.\gamma \times \alpha}, \quad \gamma' = \frac{\alpha \times \beta}{\gamma.\alpha \times \beta}. \tag{1}$$

The identical equations (4) and (8) of the preceding number thus become

$$\rho = (\rho.\alpha')\alpha + (\rho.\beta')\beta + (\rho.\gamma')\gamma, \tag{2}$$

$$\rho = (\rho.\alpha)\alpha' + (\rho.\beta)\beta' + (\rho.\gamma)\gamma'. \tag{3}$$

We may infer from the similarity of these equations that the relations of α, β, γ, and α', β', γ' are reciprocal, a proposition which is easily proved directly. For the equations

$$\alpha = \frac{\beta' \times \gamma'}{\alpha'.\beta' \times \gamma'}, \quad \beta = \frac{\gamma' \times \alpha'}{\beta'.\gamma' \times \alpha'}, \quad \gamma = \frac{\alpha' \times \beta'}{\gamma'.\alpha' \times \beta'} \tag{4}$$

are satisfied identically by the substitution of the values of α', β', and γ' given in equations (1). (See Nos. 31 and 34.)

Def.—It will be convenient to use the term reciprocal to designate these relations, i.e., we shall say that three vectors are *reciprocals* of three others, when they satisfy relations similar to those expressed in equations (1) or (4).

With this understanding we may say:—

The coefficients by which any vector is expressed in terms of three other vectors are the direct products of that vector with the reciprocals of the three.

Among other relations which are satisfied by reciprocal systems of vectors are the following:

$$\alpha.\alpha' = \beta.\beta' = \gamma.\gamma' = 1,$$

$$\alpha.\beta' = 0, \quad \alpha.\gamma' = 0, \quad \beta.\alpha' = 0, \quad \beta.\gamma' = 0, \quad \gamma.\alpha' = 0, \quad \gamma.\beta' = 0. \tag{5}$$

These nine equations may be regarded as defining the relations between α, β, γ and α', β', γ' as reciprocals.

$$(\alpha.\beta \times \gamma)(\alpha'.\beta' \times \gamma') = 1. \tag{6}$$

(See No. 34.)

$$\alpha \times \alpha' + \beta \times \beta' + \gamma \times \gamma' = 0. \tag{7}$$

(See No. 29.)

A system of three mutually perpendicular unit vectors is reciprocal to itself, and only such a system.

The identical equation

$$\rho = (\rho.i)i + (\rho.j)j + (\rho.k)k \tag{8}$$

may be regarded as a particular case of equation (2).

The system reciprocal to $\alpha \times \beta$, $\beta \times \gamma$, $\gamma \times \alpha$ is

$$\alpha' \times \beta', \quad \beta' \times \gamma', \quad \gamma' \times \alpha',$$

or

$$\frac{\alpha}{\alpha.\beta \times \gamma}, \quad \frac{\beta}{\alpha.\beta \times \gamma}, \quad \frac{\gamma}{\alpha.\beta \times \gamma}.$$

38a. If we multiply the identical equation (8) of No. 37 by $\sigma \times \tau$, we obtain the equation

$$\begin{aligned}(\alpha.\beta \times \gamma)(\rho.\sigma \times \tau) =& \alpha.\rho(\beta.\sigma\gamma.\tau - \beta.\tau\gamma.\sigma) \\ &+ \beta.\rho(\gamma.\sigma\alpha.\tau - \gamma.\tau\alpha.\sigma) + \gamma.\rho(\alpha.\sigma\beta.\tau - \alpha.\tau\beta.\sigma),\end{aligned}$$

which is therefore identical. But this equation cannot subsist identically, unless

$$\begin{aligned}(\alpha.\beta \times \gamma)\sigma \times \tau =& \alpha(\beta.\sigma\gamma.\tau - \beta.\tau\gamma.\sigma) \\ &+ \beta(\gamma.\sigma\alpha.\tau - \gamma.\tau\alpha.\sigma) + \gamma(\alpha.\sigma\beta.\tau - \alpha.\tau\beta.\sigma)\end{aligned}$$

is also an identical equation. (The reader will observe that in each of these equations the second member may be expressed as a determinant.)

From these transformations, with those already given, it follows that a product formed of any number of letters (representing vectors and scalars), combined in any possible way by scalar, direct, and skew multiplications, may be reduced to a sum of products, containing each the sign $\times$ once and only once, when the original product contains it an odd number of times, or entirely free from the sign, when the original product contains it an even number of times.

39. *Scalar equations of the first degree with respect to an unknown vector.*—It is easily shown that any scalar equation of the first degree with respect to an unknown vector ρ, in which all the other quantities are known, may be reduced to the form

$$\rho.\alpha = a,$$

in which α and a are known. (See No. 35.) Three such equations will afford the value of ρ (by equation (8) of No. 37, or equation (3) of No. 38), which may be used to eliminate ρ from any other equation either scalar or vector.

When we have four scalar equations of the first degree with respect to ρ, the elimination may be performed most symmetrically by substituting the values of $\rho.\alpha$, etc., in the equation

$$(\rho.\alpha)(\beta.\gamma \times \delta) - (\rho.\beta)(\gamma.\delta \times \alpha) + (\rho.\gamma)(\delta.\alpha \times \beta) - (\rho.\delta)(\alpha.\beta \times \gamma) = 0,$$

which is obtained from equation (8) of No. 37 by multiplying directly by δ. It may also be obtained from equation (5) of No. 37 by writing δ for ρ, and then multiplying directly by ρ.

40. *Solution of a vector equation of the first degree with respect to the unknown vector*.—It is now easy to solve an equation of the form

$$\delta = \alpha(\lambda.\rho) + \beta(\mu.\rho) + \gamma(\nu.\rho), \tag{1}$$

where α, β, γ, δ, λ, μ, and ν represent known vectors. Multiplying directly by $\beta \times \gamma$, by $\gamma \times \alpha$, and by $\alpha \times \beta$, we obtain

$$\beta.\gamma \times \delta = (\beta.\gamma \times \alpha)(\lambda.\rho), \quad \gamma.\alpha \times \delta = (\gamma.\alpha \times \beta)(\mu.\rho),$$

$$\alpha.\beta \times \delta = (\alpha.\beta \times \gamma)(\nu.\rho);$$

or
$$\alpha'.\delta = \lambda.\rho, \quad \beta'.\delta = \mu.\rho, \quad \gamma'.\delta = \nu.\rho,$$

where α', β', γ' are the reciprocals of α, β, γ. Substituting these values in the identical equation

$$\rho = \lambda'(\lambda.\rho) + \mu'(\mu.\rho) + \nu'(\nu.\rho),$$

in which λ', μ', ν' are the reciprocals of λ, μ, ν (see No. 38), we have

$$\rho = \lambda'(\alpha'.\delta) + \mu'(\beta'.\delta) + \nu'(\gamma'.\delta), \tag{2}$$

which is the solution required.

It results from the principle stated in No. 35, that any vector equation of the first degree with respect to ρ may be reduced to the form

$$\delta = \alpha(\lambda.\rho) + \beta(\mu.\rho) + \gamma(\nu.\rho) + a\rho + \epsilon \times \rho.$$

But
$$a\rho = a\lambda'(\lambda.\rho) + a\mu'(\mu.\rho) + a\nu'(\nu.\rho),$$

and
$$\epsilon \times \rho = \epsilon \times \lambda'(\lambda.\rho) + \epsilon \times \mu'(\mu.\rho) + \epsilon \times \nu'(\nu.\rho),$$

where λ', μ', ν' represent, as before, the reciprocals of λ, μ, ν. By substitution of these values the equation is reduced to the form of equation (1), which may therefore be regarded as the most general form of a vector equation of the first degree with respect to ρ.

41. *Relations between two normal systems of unit vectors.*—If i, j, k, and i', j', k' are two normal systems of unit vectors, we have

$$\left.\begin{aligned} i' &= (i.i')i + (j.i')j + (k.i')k, \\ j' &= (i,j')i + (j.j')j + (k.j')k, \\ k' &= (i.k')i + (j.k')j + (k.k')k, \end{aligned}\right\} \tag{1}$$

and

$$\left.\begin{aligned} i &= (i.i')i' + (i.j')j' + (i.k')k', \\ j &= (j.i')i' + (j.j')j' + (j.k')k', \\ k &= (k.i')i' + (k.j')j' + (k.k')k'. \end{aligned}\right\} \tag{2}$$

(See equation (8) of No. 38.)

The nine coefficients in these equations are evidently the cosines of the nine angles made by a vector of one system with a vector of the other system. The principal relations of these cosines are easily deduced. By direct multiplication of each of the preceding equations with itself, we obtain six equations of the type

$$(i.i')^2 + (j.i')^2 + (k.i')^2 = 1. \tag{3}$$

By direct multiplication of equations (1) with each other, and of equations (2) with each other, we obtain six of the type

$$(i.i')(i.j') + (j.i')(j.j') + (k.i')(k.j') = 0. \tag{4}$$

By skew multiplication of equations (1) with each other, we obtain three of the type

$$\begin{aligned} k' = \{(j.i')(k.j') - (k.i')(j.j')\}i + \{(k.i')(i.j') - (i.i')(k.j')\}j \\ + \{(i.i')(j.j') - (j.i')(i.j')\}k. \end{aligned}$$

Comparing these three equations with the original three, we obtain nine of the type

$$i.k' = (j.i')(k.j') - (k.i')(j.j'). \tag{5}$$

Finally, if we equate the scalar product of the three right hand members of (1) with that of the three left hand members, we obtain

$$\begin{aligned} (i.i')(j.j')(k.k') + (i.j')(j.k')(k.i') + (i.k')(j.i')(k.j') \\ -(k.i')(j.j')(i.k') - (k.j')(j.k')(i.i') - (k.k')(j.i')(i.j') = 1. \end{aligned} \tag{6}$$

Equations (1) and (2) (if the expressions in the parentheses are supposed replaced by numerical values) represent the linear relations which subsist between one vector of one system and the three vectors of the other system. If we desire to express the similar relations which subsist between two vectors of one system and two of the other, we may take the skew products of equations (1) with equations (2), after transposing all terms in the latter. This will afford nine equations of the type

$$(i.j')k' - (i.k')j' = (k.i')j - (j.i')k. \tag{7}$$

We may divide an equation by an indeterminate direct factor. [MS. note by author.]

Chapter 2

CONCERNING THE DIFFERENTIAL AND INTEGRAL CALCULUS OF VECTORS.

42. *Differentials of vectors*.—The *differential* of a vector is the *geometrical* difference of two values of that vector which differ infinitely little. It is itself a vector, and may make any angle with the vector differentiated. It is expressed by the same sign (d) as the differentials of ordinary analysis.

With reference to any fixed axes, the components of the differential of a vector are manifestly equal to the differentials of the components of the vector, i.e., if α, β, and γ are fixed unit vectors, and

$$\rho = x\alpha + y\beta + z\gamma,$$
$$d\rho = dx\,\alpha + dy\,\beta + dz\,\gamma.$$

43. *Differential of a function of several variables*.—The differential of a vector or scalar function of any number of vector or scalar variables is evidently the sum (geometrical or algebraic, according as the function is vector or scalar) of the differentials of the function due to the separate variation of the several variables.

44. *Differential of a product*.—The differential of a product of any kind due to the variation of a single factor is obtained by prefixing the sign of differentiation to that factor in the product. This is evidently true of differentials, since it will hold true even of finite differences.

45. From these principles we obtain the following identical equations:

$$d(\alpha + \beta) = d\alpha + d\beta, \tag{1}$$
$$d(n\alpha) = dn\,\alpha + n\,d\alpha, \tag{2}$$
$$d(\alpha.\beta) = d\alpha.\beta + \alpha.d\beta, \tag{3}$$

$$
\begin{aligned}
d[\alpha \times \beta] &= d\alpha \times \beta + \alpha \times d\beta, && (4)\\
d(\alpha.\beta \times \gamma) &= d\alpha.\beta \times \gamma + \alpha.d\beta \times \gamma + \alpha.\beta \times d\gamma, && (5)\\
d[(\alpha.\beta)\gamma] &= (d\alpha.\beta)\gamma + (\alpha.d\beta)\gamma + (\alpha.\beta)d\gamma. && (6)
\end{aligned}
$$

46. *Differential coefficient with respect to a scalar*.—The quotient obtained by dividing the differential of a vector due to the variation of any scalar of which it is a function by the differential of that scalar is called the differential coefficient of the vector with respect to the scalar, and is indicated in the same manner as the differential coefficients of ordinary analysis.

If we suppose the quantifies occurring in the six equations of the last section to be functions of a scalar t, we may substitute $\frac{d}{dt}$ for d in those equations since this is only to divide all terms by the scalar dt.

47. *Successive differentiations*.—The differential coefficient of a vector with respect to a scalar is of course a finite vector, of which we may take the differential, or the differential coefficient with respect to the same or any other scalar. We thus obtain differential coefficients of the higher orders, which are indicated as in the scalar calculus.

A few examples will serve for illustration.

If ρ is the vector drawn from a fixed origin to a moving point at any time t, $\frac{d\rho}{dt}$ will be the vector representing the velocity of the point, and $\frac{d^2\rho}{dt^2}$ the vector representing its acceleration.

If ρ is the vector drawn from a fixed origin to any point on a curve, and s the distance of that point measured on the curve from any fixed point, $\frac{d\rho}{ds}$ is a unit vector, tangent to the curve and having the direction in which s increases; $\frac{d^2\rho}{ds^2}$ is a vector directed from a point on the curve to the center of curvature, and equal to the curvature; $\frac{d\rho}{ds} \times \frac{d^2\rho}{ds^2}$ is the normal to the osculating plane, directed to the side on which the curve appears described counter-clockwise about the center of curvature, and equal to the curvature. The tortuosity (or rate of rotation of the osculating plane, considered as positive when the rotation appears counter-clockwise as seen from the direction in which s increases) is represented by

$$\frac{\frac{d\rho}{ds} \cdot \frac{d^2\rho}{ds^2} \times \frac{d^3\rho}{ds^3}}{\frac{d^2\rho}{ds^2} \cdot \frac{d^2\rho}{ds^2}}.$$

48. *Integration of an equation between differentials*.—If t and u are two single-valued continuous scalar functions of any number of scalar or vector variables, and

$$dt = du,$$

then

$$t = u + a,$$

where a is a scalar constant.

Or, if τ and ω are two single-valued continuous vector functions of any number of scalar or vector variables, and

$$d\tau = d\omega,$$

then $$\tau = \omega + \alpha,$$

where α is a vector constant.

When the above hypotheses are not satisfied in general, but will be satisfied if the variations of the independent variables are confined within certain limits, then the conclusions will hold within those limits, provided that we can pass by continuous variation of the independent variables from any values within the limits to any other values within them, without transgressing the limits.

49. So far, it will be observed, all operations have been entirely analogous to those of the ordinary calculus.

Functions of Position in Space.

50. *Def.*—If u is any scalar function of position in space (i.e., any scalar quantity having continuously varying values in space), ∇u is the vector function of position in space which has everywhere the direction of the most rapid increase of u, and a magnitude equal to the rate of that increase per unit of length. ∇u may be called the *derivative* of u, and u, the *primitive* of ∇u.

We may also take any one of the Nos. 51, 52, 53 for the definition of ∇u.

51. If ρ is the vector defining the position of a point in space,

$$du = \nabla u.d\rho.$$

52.
$$\nabla u = i\frac{du}{dx} + j\frac{du}{dy} + k\frac{du}{dz}.$$

53.
$$\frac{du}{dx} = i.\nabla u, \quad \frac{du}{dy} = j.\nabla u, \quad \frac{du}{dz} = k.\nabla u.$$

54. *Def.*—If ω is a vector having continuously varying values in space,

$$\nabla.\omega = i.\frac{d\omega}{dx} + j.\frac{d\omega}{dy} + k.\frac{d\omega}{dz}, \tag{1}$$

and
$$\nabla \times \omega = i \times \frac{d\omega}{dx} + j \times \frac{d\omega}{dy} + k \times \frac{d\omega}{dz}. \tag{2}$$

$\nabla.\omega$ is called the *divergence* of ω, and $\nabla \times \omega$ its *curl*.

If we set
$$\omega = Xi + Yj + Zk,$$

we obtain by substitution the equations

$$\nabla.\omega = \frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz}$$

and $$\nabla \times \omega = i\left(\frac{dZ}{dy} - \frac{dY}{dz}\right) + j\left(\frac{dX}{dz} - \frac{dZ}{dx}\right) + k\left(\frac{dY}{dx} - \frac{dX}{dy}\right),$$

which may also be regarded as defining $\nabla.\omega$, and $\nabla \times \omega$.

55. *Surface-integrals.*—The integral $\int\int \omega.d\sigma$, in which $d\sigma$ represents an element of some surface, is called the surface-integral of ω for that surface. It is understood here and elsewhere, when a vector is said to represent a plane surface (or an element of surface which may be regarded as plane), that the magnitude of the vector represents the area of the surface, and that the direction of the vector represents that of the normal drawn toward the positive side of the surface. When the surface is defined as the boundary of a certain space, the outside of the surface is regarded as positive.

The surface-integral of any given space (i.e., the surface-integral of the surface bounding that space) is evidently equal to the sum of the surface-integrals of all the parts into which the original space may be divided. For the integrals relating to the surfaces dividing the parts will evidently cancel in such a sum.

The surface-integral of ω for a closed surface bounding a space dv infinitely small in all its dimensions is

$$\nabla.\omega\, dv.$$

This follows immediately from the definition of $\nabla\omega$, when the space is a parallelopiped bounded by planes perpendicular to i, j, k. In other cases, we may imagine the space—or rather a space nearly coincident with the given space and of the same volume dv—to be divided up into such parallelopipeds. The surface-integral for the space made up of the parallelopipeds will be the sum of the surface-integrals of all the parallelopipeds, and will therefore be expressed by $\nabla.\omega\, dv$. The surface-integral of the original space will have sensibly the same value, and will therefore be represented by the same formula. It follows that the value of $\nabla.\omega$ does not depend upon the system of unit vectors employed in its definition.

It is possible to attribute such a physical signification to the quantities concerned in the above proposition, as shall make it evident almost without demonstration. Let us suppose ω to represent a flux of any substance. The rate of decrease of the density of that substance at any point will be obtained by dividing the surface-integral of the flux for any infinitely small closed surface about the point by the volume enclosed. This quotient must therefore be independent of the form of the surface. We may define $\nabla.\omega$ as representing that quotient, and then obtain equation (1) of No. 54 by applying the general principle to the case of the rectangular parallelopiped.

56. *Skew surface-integrals.*—The integral $\int\int d\sigma \times \omega$ may be called the skew surface-integral of ω. It is evidently a vector. For a closed surface bounding a space dv infinitely small in all dimensions this integral reduces to $\nabla \times \omega\, dv$, as is easily shown by reasoning like that of No. 55.

57. *Integration.*—If dv represents an element of any space, and $d\sigma$ an element of the bounding surface,

$$\int\int\int \nabla.\omega\, dv = \int\int \omega.d\sigma.$$

For the first member of this equation represents the sum of the surface-integrals of all the elements of the given space. We may regard this principle as affording a means of integration, since we may use it to reduce a triple integral (of a certain form) to a double integral.

The principle may also be expressed as follows:

The surface-integral of any vector function of position in space for a closed surface is equal to the volume-integral of the divergence of that function for the space enclosed.

58. *Line-integrals*.—The integral $\int \omega.d\rho$, in which $d\rho$ denotes the element of a line, is called the *line-integral* of ω for that line. It is implied that one of the directions of the line is distinguished as positive. When the line is regarded as bounding a surface, that side of the surface will always be regarded as positive, on which the surface appears to be circumscribed counter-clockwise.

59. *Integration*.—From No. 51 we obtain directly

$$\int \nabla u.d\rho = u'' - u',$$

where the single and double accents distinguish the values relating to the beginning and end of the line.

In other words,—The line-integral of the derivative of any (continuous and single-valued) scalar function of position in space is equal to the difference of the values of the function at the extremities of the line. For a closed line the integral vanishes.

60. *Integration*.—The following principle may be used to reduce double integrals of a certain form to simple integrals.

If $d\sigma$ represents an element of any surface, and $d\rho$ an element of the bounding line,

$$\int\int \nabla \times \omega.d\sigma = \int \omega.d\rho.$$

In other words,—The line-integral of any vector function of position in space for a closed line is equal to the surface-integral of the curl of that function for any surface bounded by the line.

To prove this principle, we will consider the variation of the line-integral which is due to a variation in the closed line for which the integral is taken. We have, in the first place,

$$\delta \int \omega.d\rho = \int \delta\omega.d\rho + \int \omega.\delta\, d\rho.$$

But $$\omega.\delta\, d\rho = d(\omega.\delta\rho) - d\omega.\delta\rho.$$

Therefore, since $\int d(\omega.\delta\rho) = 0$ for a closed line,

$$\delta \int \omega.d\rho = \int \delta\omega.d\rho - \int d\omega.\delta\rho.$$

Now $$\delta\omega = \Sigma \left[\frac{d\omega}{dx} \delta x \right] = \Sigma \left[\frac{d\omega}{dx} (i.\delta\rho) \right],$$

and $$d\omega = \Sigma \left[\frac{d\omega}{dx} dx \right] = \Sigma \left[\frac{d\omega}{dx} (i.d\rho) \right],$$

where the summation relates to the coordinate axes and connected quantities. Substituting these values in the preceding equation, we get

$$\delta \int \omega . d\rho = \int \Sigma \left((i.\delta\rho) \left(\frac{d\omega}{dx} . d\rho \right) - (i.d\rho) \left(\frac{d\omega}{dx} . \delta\rho \right) \right),$$

or by No. 30,

$$\delta \int \omega . d\rho = \int \Sigma \left[i \times \frac{d\omega}{dx} \right] . [\delta\rho \times d\rho] = \int \nabla \times \omega . [\delta\rho \times d\rho].$$

But $\delta\rho \times d\rho$ represents an element of the surface generated by the motion of the element $d\rho$, and the last member of the equation is the surface-integral of $\nabla \times \omega$, for the infinitesimal surface generated by the motion of the whole line. Hence, if we conceive of a closed curve passing gradually from an infinitesimal loop to any finite form, the differential of the line-integral of ω for that curve will be equal to the differential of the surface integral of $\nabla \times \omega$ for the surface generated: therefore, since both integrals commence with the value zero, they must always be equal to each other. Such a mode of generation will evidently apply to any surface closing any loop.

61. The line-integral of ω, for a closed line bounding a plane surface $d\sigma$ infinitely small in all its dimensions is therefore

$$\nabla \times \omega . d\sigma.$$

This principle affords a definition of $\nabla \times \omega$ which is independent of any reference to coordinate axes. If we imagine a circle described about a fixed point to vary its orientation while keeping the same size, there will be a certain position of the circle for which the line-integral of ω will be a maximum, unless the line-integral vanishes for all positions of the circle. The axis of the circle in this position, drawn toward the side on which a positive motion in the circle appears counter-clockwise, gives the direction of $\nabla \times \omega$, and the quotient of the integral divided by the area of the circle gives the magnitude of $\nabla \times \omega$.

∇, $\nabla .$, and $\nabla\times$ applied to Functions of Functions of Position.

62. A constant scalar factor after ∇, $\nabla .$, or $\nabla\times$ may be placed before the symbol.

68. If $f(u)$ denotes any scalar function of u, and $f'(u)$ the derived function,

$$\nabla f(u) = f'(u) \nabla u.$$

64. If u or ω is a function of several scalar or vector variables which are themselves functions of the position of a single point, the value of ∇u or $\nabla . \omega$ or $\nabla \times \omega$, will be equal to the sum of the values obtained by making successively all but each one of these variables constant.

65. By the use of this principle we easily derive the following identical equations:

$$\nabla(t + u) = \nabla t + \nabla u. \tag{1}$$

$$\nabla.(\tau + \omega) = \nabla.\tau + \nabla.\omega. \quad \nabla \times [\tau + \omega] = \nabla \times \tau + \nabla \times \omega. \tag{2}$$

$$\nabla(tu) = u\nabla t + t\nabla u. \tag{3}$$
$$\nabla.(u\omega) = \omega.\nabla u + u\nabla.\omega. \tag{4}$$
$$\nabla \times [u\omega] = u\nabla \times \omega - \omega \times \nabla u. \tag{5}$$
$$\nabla.[\tau \times \omega] = \omega.\nabla \times \tau - \tau.\nabla \times \omega. \tag{6}$$

The student will observe an analogy between these equations and the formulæof multiplication. (In the last four equations the analogy appears most distinctly when we regard all the factors but one as constant.) Some of the more curious features of this analogy are due to the fact that the ∇ contains implicitly the vectors i, j and k, which are to be multiplied into the following quantities.

Combinations of the Operators ∇, $\nabla.$, and $\nabla\times$.

66. If u is any scalar function of position in space,

$$\nabla \times \nabla u = 0,$$

as may be derived directly from the definitions of these operators.

67. Conversely, if ω is such a vector function of position in space that

$$\nabla \times \omega = 0,$$

ω is the derivative of a scalar function of position in space. This will appear from the following considerations:

The line-integral $\int \omega.d\rho$ will vanish for any closed line, since it may be expressed as the surface-integral of $\nabla \times \omega$. (No. 60.) The line-integral taken from one given point P' to another given point P'' is independent of the line between the points for which the integral is taken. (For, if two lines joining the same points gave different values, by reversing one we should obtain a closed line for which the integral would not vanish.) If we set u equal to this line-integral, supposing P'' to be variable and P' to be constant in position, u will be a scalar function of the position of the point P'', satisfying the condition $du = \omega.d\rho$, or, by No. 51, $\nabla u = \omega$. There will evidently be an infinite number of functions satisfying this condition, which will differ from one another by constant quantities.

If the region for which $\nabla \times \omega = 0$ is unlimited, these functions will be single-valued. If the region is limited, but acyclic,[1] the functions will still be single-valued

[1] If every closed line within a given region can contract to a single point without breaking its continuity, or passing oct of the region, the region is called *acyclic*, otherwise *cyclic*.

A cyclic region may be made acyclic by diaphragms, which must then be regarded as forming part of the surface bounding the region, each diaphragm contributing its own area twice to that surface. This process may be used to reduce many-valued functions of position in space, having single-valued derivatives, to single-valued functions.

When functions are mentioned or implied in the notation, the reader will always understand single-valued functions, unless the contrary is distinctly intimated, or the case is one in which the distinction is obviously immaterial. Diaphragms may be applied to bring functions naturally many-valued under the application of some of the following theorems, as Nos. 74 ff.

and satisfy the condition $\nabla u = \omega$ within the same region. If the region is cyclic, we may determine functions satisfying the condition $\nabla u = \omega$ within the region, but they will not necessarily be single-valued.

68. If ω is any vector function of position in space, $\nabla.\nabla \times \omega = 0$. This may be deduced directly from the definitions of No. 54.

The converse of this proposition will be proved hereafter.

69. If u is any scalar function of position in space, we have by Nos. 52 and 54

$$\nabla.\nabla u = \left(\frac{d^2}{d^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right) u.$$

70. *Def.*—If ω is any vector function of position in space, we may define $\nabla.\nabla\omega$ by the equation

$$\nabla.\nabla\omega = \left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right) \omega,$$

the expression $\nabla.\nabla$ being regarded, for the present at least, as a single operator when applied to a vector. (It will be remembered that no meaning has been attributed to ∇ before a vector.) It should be noticed that if

$$\omega = iX + jY + kZ,$$

$$\nabla.\nabla\omega = i\nabla.\nabla X + j\nabla.\nabla Y + k\nabla.\nabla Z,$$

that is, the operator $\nabla.\nabla$ applied to a vector affects separately its scalar components.

71. From the above definition with those of Nos. 52 and 54 we may easily obtain

$$\nabla.\nabla\omega = \nabla\nabla.\omega - \nabla \times \nabla \times \omega.$$

The effect of the operator $\nabla.\nabla$ is therefore independent of the directions of the axes used in its definition.

72. The expression $-\frac{1}{6}a^2\nabla.\nabla u$, where a is any infinitesimal scalar, evidently represents the excess of the value of the scalar function u at the point considered above the average of its values at six points at the following vector distances: ai, $-ai$, aj, $-aj$, ak, $-ak$. Since the directions of i, j, and k are immaterial (provided that they are at right angles to each other), the excess of the value of u at the central point above its average value in a spherical surface of radius a constructed about that point as the center will be represented by the same expression, $-\frac{1}{6}a^2\nabla.\nabla u$.

Precisely the same is true of a vector function, if it is understood that the additions and subtractions implied in the terms *average* and *excess* are geometrical additions and subtractions.

Maxwell has called $-\nabla.\nabla u$ the *concentration* of u, whether u is scalar or vector. We may call $\nabla.\nabla u$ (or $\nabla.\nabla\omega$), which is proportioned to the excess of the average value of the function in an infinitesimal spherical surface above the value at the center, the *dispersion* of u (or ω).

Transformation of Definite Integrals.

73. From the equations of No. 65, with the principles of integration of Nos. 57, 59, and 60, we may deduce various transformations of definite integrals, which are entirely analogous to those known in the scalar calculus under the name of *integration by parts*. The following formulæ(like those of Nos. 57, 59, and 60) are written for the case of continuous values of the quantities (scalar and vector) to which the signs ∇, $\nabla.$, and $\nabla\times$ are applied. It is left to the student to complete the formulæfor cases of discontinuity in these values. The manner in which this is to be done may in each case be inferred from the nature of the formula itself. The most important discontinuities of scalars are those which occur at surfaces: in the case of vectors discontinuities at surfaces, at lines, and at points, should be considered.

74. From equation (3) we obtain

$$\int \nabla(tu).d\rho = t''u'' - t'u' = \int u\nabla t.d\rho + \int t\nabla u.d\rho,$$

where the accents distinguish the quantities relating to the limits of the line-integrals. We are thus able to reduce a line-integral of the form $\int u\nabla t.d\rho$ to the form $-\int t\nabla u.d\rho$ with quantities free from the sign of integration.

75. From equation (5) we obtain

$$\int\int \nabla\times(u\omega).d\sigma = \int u\omega.d\rho = \int\int u\nabla\times\omega.d\rho - \int\int \omega\times\nabla u.d\sigma,$$

where, as elsewhere in these equations, the line-integral relates to the boundary of the surface-integral.

From this, by substitution of ∇t for ω, we may derive as a particular case

$$\int\int \nabla u\times\nabla t.d\sigma = \int u\nabla t.d\rho = -\int t\nabla u.d\rho.$$

76. From equation (4) we obtain

$$\int\int\int \nabla.[u\omega]dv = \int\int u\omega.d\sigma = \int\int\int \omega.\nabla u\, dv + \int\int\int u\nabla.\omega\, dv,$$

where, as elsewhere in these equations, the surface-integral relates the boundary of the volume-integrals.

From this, by substitution of ∇t for ω, we derive as a particular case

$$\begin{aligned}\int\int\int \nabla t.\nabla u\, dv &= \int\int u\nabla t.d\sigma - \int\int\int u\nabla.\nabla t\, dv \\ &= \int\int t\nabla u.d\sigma - \int\int\int t\nabla.\nabla u\, dv,\end{aligned}$$

which is Green's Theorem. The substitution of $s\nabla t$ for ω gives the more general form of this theorem which is due to Thomson, viz.,

$$\begin{aligned}\int\int\int s\nabla t.\nabla u\, dv &= \int\int us\nabla t.d\sigma - \int\int\int u\nabla.[s\nabla t]dv \\ &= \int\int ts\nabla u.d\sigma - \int\int\int t\nabla.[s\nabla u]dv.\end{aligned}$$

77. From equation (6) we obtain

$$\iiint \nabla.[\tau \times \omega]dv = \iint \tau \times \omega.d\sigma$$
$$= \iiint \omega.\nabla \times \tau\, dv - \iiint \tau.\nabla \times \omega\, dv.$$

A particular case is

$$\iiint \nabla u.\nabla \times \omega\, dv = \iint \omega \times \nabla u.d\sigma.$$

Integration of Differential Equations.

78. If throughout any continuous space (or in all space)

$$\nabla u = 0,$$

then throughout the same space

$$u = \text{constant}.$$

79. If throughout any continuous space (or in all space)

$$\nabla.\nabla u = 0,$$

and in any finite part of that space, or in any finite surface in or bounding it,

$$\nabla u = 0,$$

then throughout the whole space

$$\nabla u = 0, \quad \text{and} \quad u = \text{constant}.$$

This will appear from the following considerations:

If $\nabla u = 0$ in any finite part of the space, u is constant in that part. If u is not constant throughout, let us imagine a sphere situated principally in the part in which u is constant, but projecting slightly into a part in which u has a greater value, or else into a part in which u has a less. The surface-integral of ∇u for the part of the spherical surface in the region where u is constant will have the value zero: for the other part of the surface, the integral will be either greater than zero, or less than zero. Therefore the whole surface-integral for the spherical surface will not have the value zero, which is required by the general condition, $\nabla.\nabla u = 0$.

Again, if $\nabla u = 0$ only in a surface in or bounding the space in which $\nabla.\nabla u = 0$, u will be constant in this surface, and the surface will be contiguous to a region in which $\nabla.\nabla u = 0$ and u has a greater value than in the surface, or else a less value than in the surface. Let us imagine a sphere lying principally on the other side of the surface, but projecting slightly into this region, and let us particularly consider the surface-integral of ∇u for the small segment cut off by the surface $\nabla u = 0$. The integral for that part of the surface of the segment which consists of part of the surface $\nabla u = 0$ will have the value zero, the integral for the spherical part will have a value either greater than zero or else less than zero. Therefore the integral for the whole surface of the segment cannot have the value zero, which is demanded by the general condition, $\nabla.\nabla u = 0$.

80. If throughout a certain space (which need not be continuous, and which may extend to infinity)

$$\nabla.\nabla u = 0,$$

and in all the bounding surfaces

$$u = \text{constant} = a,$$

and (in case the space extends to infinity) if at infinite distances within the space $u = a$,—then throughout the space

$$\nabla u = 0, \quad \text{and} \quad u = a.$$

For, if anywhere in the interior of the space ∇u has a value different from zero, we may find a point P where such is the case, and where u has a value b different from a,—to fix our ideas we will say *less*. Imagine a surface enclosing all of the space in which $u < b$. (This must be possible, since that part of the space does not reach to infinity.) The surface-integral of ∇u for this surface has the value zero in virtue of the general condition $\nabla.\nabla u = 0$. But, from the manner in which the surface is defined, no part of the integral can be negative. Therefore no part of the integral can be positive, and the supposition made with respect to the point P is untenable. That the supposition that $b > a$ is untenable may be shown in a similar manner. Therefore the value of u is constant.

This proposition may be generalized by substituting the condition $\nabla.[t\nabla u] = 0$ for $\nabla.\nabla u = 0$, t denoting any positive (or any negative) scalar function of position in space. The conclusion would be the same, and the demonstration similar.

81. If throughout a certain space (which need not be continuous, and which may extend to infinity)

$$\nabla.\nabla u = 0,$$

and in all the bounding surfaces the normal component of ∇u vanishes, and at infinite distances within the space (if such there are) $r^2 \frac{du}{dr} = 0$, where r denotes the distance from a fixed origin, then throughout the space

$$\nabla u = 0,$$

and in each continuous portion of the same

$$u = \text{constant}.$$

For, if anywhere in the space in question ∇u has a value different from zero, let it have such a value at a point P, and let a be there equal to b. Imagine a spherical surface about the above-mentioned origin as center, enclosing the point P, and with a radius r. Consider that portion of the space to which the theorem relates which is within the sphere and in which $u < b$. The surface integral of ∇u for this space is equal to zero in virtue of the general condition $\nabla.\nabla u = 0$. That part of the integral (if any) which relates to a portion of the spherical surface has a value numerically not greater than $4\pi r^2 \left(\frac{du}{dr}\right)'$, where $\left(\frac{du}{dr}\right)'$ denotes the greatest numerical value of $\frac{du}{dr}$ in the portion of the spherical surface considered. Hence, the value of this part of the surface-integral may be made less (numerically) than any assignable quantity by giving to r a sufficiently great value. Hence, the other part of the surface-integral (viz., that relating to the surface in which $u = b$, and to the boundary of the space to which the theorem relates) may be given a value differing from zero by less than any assignable quantity. But no part of the integral relating to this surface can be negative. Therefore no part can be positive, and the supposition relative to the point P is untenable.

This proposition also may be generalized by substituting $\nabla.[t\nabla u] = 0$ for $\nabla.\nabla u = 0$, and $tr^2 \frac{du}{dr} = 0$ for $r^2 \frac{du}{dr} = 0$.

82. If throughout any continuous space (or in all space)

$$\nabla t = \nabla u,$$

then throughout the same space

$$t = u + \text{const.}$$

The truth of this and the three following theorems will be apparent if we consider the difference $t - u$.

83. If throughout any continuous space (or in all space)

$$\nabla.\nabla t = \nabla.\nabla u,$$

and in any finite part of that space, or in any finite surface in or bounding it,

$$\nabla t = \nabla u,$$

then throughout the whole space

$$\nabla t = \nabla u, \quad \text{and} \quad t = u + \text{const.}$$

84. If throughout a certain space (which need not be continuous, and which may extend to infinity)

$$\nabla.\nabla t = \nabla.\nabla u,$$

and in all the bounding surfaces

$$t = u,$$

and at infinite distances within the space (if such there are)

$$t = u,$$

then throughout the space

$$t = u.$$

85. If throughout a certain space (which need not be continuous, and which may extend to infinity)

$$\nabla.\nabla t = \nabla.\nabla u,$$

and in all the bounding surfaces the normal components of ∇t and ∇u are equal, and at infinite distances within the space (if such there are) $r^2\left(\frac{dt}{dr} - \frac{du}{dr}\right) = 0$, where r denotes the distance from some fixed origin,—then throughout the space

$$\nabla t = \nabla u,$$

and in each continuous part of which the space consists

$$t - u = \text{constant}.$$

86. If throughout any continuous space (or in all space)

$$\nabla \times \tau = \nabla \times \omega \quad \text{and} \quad \nabla.\tau = \nabla.\omega,$$

and in any finite part of that space, or in any finite surface in or bounding it,

$$\tau = \omega,$$

then throughout the whole space

$$\tau = \omega.$$

For, since $\nabla \times (\tau - \omega) = 0$, we may set $\nabla u = \tau - \omega$, making the space acyclic (if necessary) by diaphragms. Then in the whole space u is single-valued and $\nabla.\nabla u = 0$, and in a part of the space, or in a surface in or bounding it, $\nabla u = 0$. Hence throughout the space $\nabla u = \tau - \omega = 0$.

87. If throughout an aperiphractic[2] space contained within finite boundaries but not necessarily continuous

$$\nabla \times \tau = \nabla \times \omega \quad \text{and} \quad \nabla.\tau = \nabla.\omega,$$

and in all the bounding surfaces the tangential components of τ and ω are equal, then throughout the space

$$\tau = \omega.$$

It is evidently sufficient to prove this proposition for a continuous space. Setting $\nabla u = \tau - \omega$, we have $\nabla.\nabla u = 0$ for the whole space, and $u =$ constant for its boundary, which will be a single surface for a continuous aperiphractic space. Hence throughout the space

$$\nabla u = \tau - \omega = 0.$$

88. If throughout an acyclic space contained within finite boundaries but not necessarily continuous

$$\nabla \times \tau = \nabla \times \omega \quad \text{and} \quad \nabla.\tau = \nabla.\omega,$$

[2]If a space encloses within itself another space, it is called *periphractic*, otherwise *aperiphractic*.

and in all the bounding surfaces the normal components of τ and ω are equal, then throughout the whole space

$$\tau = \omega.$$

Setting $\nabla u = \tau - \omega$, we have $\nabla.\nabla u = 0$ throughout the space, and the normal component of ∇u at the boundary equal to zero. Hence throughout the whole space $\nabla u = \tau - \omega = 0$.

89. If throughout a certain space (which need not be continuous, and which may extend to infinity)

$$\nabla.\nabla\tau = \nabla.\nabla\omega$$

and in all the bounding surfaces

$$\tau = \omega,$$

and at infinite distances within the space (if such there are)

$$\tau = \omega,$$

then throughout the whole space

$$\tau = \omega.$$

This will be apparent if we consider separately each of the scalar components of τ and ω.

Minimum Values of the Volume-integral $\int\int\int u\omega.\omega\, dv$. (Thomson's Theorems.)

90. Let it be required to determine for a certain space a vector function of position ω subject to certain conditions (to be specified hereafter), so that the volume-integral

$$\int\int\int u\omega.\omega\, dv$$

for that space shall have a minimum value, u denoting a given positive scalar function of position.

a. In the first place, let the vector ω be subject to the conditions that $\nabla.\omega$ is given within the space, and that the normal component of ω is given for the bounding surface. (This component must of course be such that the surface-integral of ω shall be equal to the volume-integral $\int \nabla.\omega\, dv$. If the space is not continuous, this must be true of each continuous portion of it. See No. 57.) The solution is that $\nabla \times (u\omega) = 0$, or more generally, that the line-integral of $u\omega$ for any closed curve in the space shall vanish.

The existence of the minimum requires that

$$\int\int\int u\omega.\delta\omega\, dv = 0,$$

while $\delta\omega$ is subject to the limitation that

$$\nabla.\delta\omega = 0,$$

and that the normal component of $\delta\omega$ at the bounding surface vanishes. To prove that the line-integral of $u\omega$ vanishes for any closed curve within the space, let us imagine the curve to be surrounded by an infinitely slender tube of normal section dz, which may be either constant or variable. We may satisfy the equation $\nabla.\delta\omega = 0$ by making $\delta\omega = 0$ outside of the tube, and $\delta\omega\, dz = \delta a \frac{d\rho}{ds}$ within it, δa denoting an arbitrary infinitesimal constant, ρ the position-vector, and ds an element of the length of the tube or closed curve. We have then

$$\int\int\int u\omega.\delta\omega\, dv = \int u\omega.\delta\omega\, dz\, ds = \int u\omega.d\rho\, \delta a = \delta a \int u\omega.d\rho = 0,$$

whence

$$\int u\omega.d\rho = 0. \qquad \text{Q.E.D.}$$

We may express this result by saying that $u\omega$ is the derivative of a single-valued scalar function of position in space. (See No. 67.)

If for certain parts of the surface the normal component of ω is not given for each point, but only the surface-integral of ω for each such part, then the above reasoning will apply not only to closed curves, but also to curves commencing and ending in such a part of the surface. The primitive of $u\omega$ will then have a constant value in each such part.

If the space extends to infinity and there is no special condition respecting the value of ω at infinite distances, the primitive of $u\omega$ will have a constant value at infinite distances within the space or within each separate continuous part of it.

If we except those cases in which the problem has no definite meaning because the data are such that the integral $\int u\omega.\omega\, dv$ must be infinite, it is evident that a minimum must always exist, and (on account of the quadratic form of the integral) that it is unique. That the conditions just found are sufficient to insure this minimum, is evident from the consideration that any allowable values of $\delta\omega$ may be made up of such values as we have supposed. Therefore, there will be one and only one vector function of position in space which satisfies these conditions together with those enumerated at the beginning of this number.

b. In the second place, let the vector ω be subject to the conditions that $\nabla \times \omega$ is given throughout the space, and that the tangential component of ω is given at the bounding surface. The solution is that

$$\nabla.[u\omega] = 0,$$

and, if the space is periphractic, that the surface-integral of $u\omega$ vanishes for each of the bounding surfaces.

The existence of the minimum requires that

$$\int\int\int u\omega.\delta\omega\, dv = 0,$$

while $\delta\omega$ is subject to the conditions that

$$\nabla \times \delta\omega = 0,$$

and that the tangential component of $\delta\omega$ in the bounding surface vanishes. In virtue of these conditions we may set

$$\delta\omega = \nabla\delta q,$$

where δq is an arbitrary infinitesimal scalar function of position, subject only to the condition that it is constant in each of the bounding surfaces. (See No. 67.) By substitution of this value we obtain

$$\int\int\int u\omega.\nabla\delta q\, dv = 0,$$

or integrating by parts (No. 76)

$$\int\int u\omega.d\sigma\, \delta q - \int\int\int \nabla.[u\omega]\delta q\, dv = 0.$$

Since δq is arbitrary in the volume-integral, we have throughout the whole space

$$\nabla.[u\omega] = 0;$$

and since δq has an arbitrary constant value in each of the bounding surfaces (if the boundary of the space consists of separate parts), we have for each such part

$$\int\int u\omega.d\sigma = 0.$$

Potentials, Newtonians, Laplacians.

91. *Def.*—If u' is the scalar quantity of something situated at a certain point ρ', the *potential* of u' for any point ρ is a scalar function of ρ, defined by the equation

$$\mathrm{pot}\, u' = \frac{u'}{[\rho' - \rho]_0},$$

and the Newtonian of u' for any point ρ is a vector function of ρ defined by the equation

$$\mathrm{new}\, u' = \frac{\rho' - \rho}{[\rho' - \rho]_0^3} u'.$$

Again, if ω' is the vector representing the quantity and direction of something situated at the point ρ', the potential and the Laplacian of ω' for any point ρ are vector functions of ρ defined by the equations

$$\mathrm{pot}\, \omega' = \frac{\omega'}{[\rho' - \rho]_0},$$

$$\mathrm{lap}\, \omega' = \frac{\rho' - \rho}{[\rho' - \rho]_0^3} \times \omega'.$$

92. If u or ω, is a scalar or vector function of position in space, we may write $\operatorname{Pot} u$, $\operatorname{New} u$, $\operatorname{Pot} \omega$, $\operatorname{Lap} \omega$, for the volume-integrals of $\operatorname{pot} u'$, etc., taken as functions of ρ'; i.e., we may set

$$\begin{aligned}
\operatorname{Pot} u &= \int\!\!\int\!\!\int \operatorname{pot} u'\, dv' = \int\!\!\int\!\!\int \frac{u'}{[\rho' - \rho]_0}\, dv', \\
\operatorname{New} u &= \int\!\!\int\!\!\int \operatorname{new} u'\, dv' = \int\!\!\int\!\!\int \frac{\rho' - \rho}{[\rho' - \rho]_0^3} u'\, dv', \\
\operatorname{Pot} \omega &= \int\!\!\int\!\!\int \operatorname{pot} \omega'\, dv' = \int\!\!\int\!\!\int \frac{\omega}{[\rho' - \rho]_0}\, dv', \\
\operatorname{Lap} \omega &= \int\!\!\int\!\!\int \operatorname{lap} \omega'\, dv' = \int\!\!\int\!\!\int \frac{\rho' - \rho}{[\rho' - \rho]_0^3} \times \omega'\, dv',
\end{aligned}$$

where the ρ is to be regarded as constant in the integration. This extends over all space, or wherever the u' or ω' have any values other than zero. These integrals may themselves be called (integral) potentials, Newtonians, and Laplacians.

93.
$$\frac{d \operatorname{Pot} u}{dx} = \operatorname{Pot} \frac{du}{dx}, \qquad \frac{d \operatorname{Pot} \omega}{dx} = \operatorname{Pot} \frac{d\omega}{dx}.$$

This will be evident with respect both to scalar and to vector functions, if we suppose that when we differentiate the potential with respect to x (thus varying the position of the point for which the potential is taken) each element of volume dv' in the implied integral remains fixed, *not in absolute position*, but in position relative to the point for which the potential is taken. This supposition is evidently allowable whenever the integration indicated by the symbol Pot tends to a definite limit when the limits of integration are indefinitely extended.

Since we may substitute y and z for x in the preceding formula, and since a constant factor of any kind may be introduced under the sign of integration, we have

$$\begin{aligned}
\nabla \operatorname{Pot} u &= \operatorname{Pot} \nabla u, \\
\nabla . \operatorname{Pot} \omega &= \operatorname{Pot} \nabla . \omega, \\
\nabla \times \operatorname{Pot} \omega &= \operatorname{Pot} \nabla \times \omega, \\
\nabla . \nabla \operatorname{Pot} u &= \operatorname{Pot} \nabla . \nabla u, \\
\nabla . \nabla \operatorname{Pot} \omega &= \operatorname{Pot} \nabla . \nabla \omega,
\end{aligned}$$

i.e., the symbols ∇, $\nabla.$, $\nabla\times$, $\nabla.\nabla$ may be applied indifferently before or after the sign Pot.

Yet a certain restriction is to be observed. When the operation of taking the (integral) potential does not give a definite finite value, the first members of these equations are to be regarded as entirely indeterminate, but the second members may have perfectly definite values. This would be the case, for example, if u or ω had a constant value throughout all space. It might seem harmless to set an indefinite expression equal to a definite, but it would be dangerous, since we might with equal right set the indefinite expression equal to other definite expressions, and then be misled into supposing these definite expressions to be equal to one another. It will be safe to say that the above equations will hold, *provided that the potential of* u *or* ω *has a definite value*. It will be observed that whenever $\mathrm{Pot}\, u$ or $\mathrm{Pot}\, \omega$ has a definite value *in general* (i.e., with the possible exception of certain points, lines, and surfaces),[3] the first members of all these equations will have definite values in general, and therefore the second members of the equation, being necessarily equal to the first, members, when these have definite values, will also have definite values in general.

94. Again, whenever $\mathrm{Pot}\, u$ has a definite value we may write

$$\nabla\, \mathrm{Pot}\, u = \nabla \int\int\int \frac{u'}{r}\, dv' = \int\int\int \nabla \frac{1}{r} u'\, dv',$$

where r stands for $[\rho' - \rho]_0$. But

$$\nabla \frac{1}{r} = \frac{\rho' - \rho}{r^3},$$

whence
$$\nabla\, \mathrm{Pot}\, u = \mathrm{New}\, u.$$

Moreover, $\mathrm{New}\, u$ will in general have a definite value, if $\mathrm{Pot}\, u$ has.

95. In like manner, whenever $\mathrm{Pot}\, \omega$ has a definite value,

$$\nabla \times \mathrm{Pot}\, \omega = \nabla \times \int\int\int \frac{\omega'}{r}\, dv' = \int\int\int \nabla \times \frac{\omega'}{r}\, dv' = \int\int\int \nabla \frac{1}{r} \times \omega'\, dv'.$$

Substituting the value of $\nabla \frac{1}{r}$ given above we have

$$\nabla \times \mathrm{Pot}\, \omega = \mathrm{Lap}\, \omega.$$

$\mathrm{Lap}\, \omega$ will have a definite value in general whenever $\mathrm{Pot}\, \omega$ has.

96. Hence, with the aid of No. 93, we obtain

$$\nabla \times \mathrm{Lap}\, \omega = \mathrm{Lap}\, \nabla \times \omega,$$
$$\nabla . \mathrm{Lap}\, \omega = 0,$$

whenever $\mathrm{Pot}\, \omega$ has a definite value.

97. By the method of No. 93 we obtain

$$\nabla . \mathrm{New}\, u = \nabla . \int\int\int \frac{\rho' - \rho}{r^3} u'\, dv' = \int\int\int \nabla u' . \frac{\rho' - \rho}{r^3}\, dv'.$$

To find the value of this integral, we may regard the point ρ, which is constant in the integration, as the center of polar coordinates. Then r becomes the radius vector of the point ρ', and we may set

$$dv' = r^2 dq\, dr,$$

[3]Whenever it is said that a function of position in space has a definite value *in general*, this phrase is to be understood as explained above. The term definite is intended to exclude both indeterminate and infinite values.

where $r^2\,dq$ is the element of a spherical surface having center at ρ and radius r. We may also set

$$\nabla u'.\frac{\rho'-\rho}{r}=\frac{du'}{dr}.$$

We thus obtain

$$\nabla.\,\mathrm{New}\,u=\int\int\int\frac{du'}{dr}dq\,dr=4\pi\int\frac{d\overline{u}'}{dr}\,dr=4\pi\overline{u}'_{r=\infty}-4\pi\overline{u}'_{r=0},$$

where $\overline{u}$ denotes the average value of u in a spherical surface of radius r about the point ρ as center.

Now if $\mathrm{Pot}\,u$ has in general a definite value, we must have $\overline{u}'=0$ for $r=\infty$. Also, $\nabla.\,\mathrm{New}\,u$ will have in general a definite value. For $r=0$, the value of $\overline{u}'$ is evidently u. We have, therefore,

$$\nabla.\,\mathrm{New}\,u=-4\pi u,$$
$$\nabla.\nabla\,\mathrm{Pot}\,u=-4\pi u.^4$$

98. If $\mathrm{Pot}\,\omega$ has in general a definite value,

$$\begin{aligned}\nabla.\nabla\,\mathrm{Pot}\,\omega&=\nabla.\nabla\,\mathrm{Pot}[ui+vj+wk]\\&=\nabla.\nabla\,\mathrm{Pot}\,ui+\nabla.\nabla\,\mathrm{Pot}\,vj+\nabla.\nabla\,\mathrm{Pot}\,wk\\&=-4\pi ui-4\pi vj-4\pi wk\\&=-4\pi\omega.\end{aligned}$$

Hence, by No. 71,

$$\nabla\times\nabla\times\mathrm{Pot}\,\omega-\nabla\nabla.\,\mathrm{Pot}\,\omega=4\pi\omega.$$

That is, $$\mathrm{Lap}\,\nabla\times\omega-\mathrm{New}\,\nabla.\omega=4\pi\omega.$$

If we set

$$\omega_1=\frac{1}{4\pi}\,\mathrm{Lap}\,\nabla\times\omega,\qquad\omega_2=\frac{-1}{4\pi}\,\mathrm{New}\,\nabla.\omega,$$

we have $$\omega=\omega_1+\omega_2$$

[4] Better thus:

$$\begin{aligned}\nabla.\nabla\,\mathrm{Pot}\,u&=\int\int\int\frac{1}{r}\nabla.\nabla u\,dv\\&=\int\int\int\nabla.\left(\frac{1}{r}\nabla u\right)\,dv-\int\int\int\nabla.\left(u\nabla\frac{1}{r}\right)\,dv+\int\int\int u\nabla.\nabla\frac{1}{r}\,dv\\&=-\int\int u\nabla\frac{1}{r}.d\sigma\\&=-4\pi u.\end{aligned}$$

[MS. note by author.]

where ω_1 and ω_2 are such functions of position that $\nabla.\omega_1 = 0$, and $\nabla \times \omega_2 = 0$. This is expressed by saying that ω_1 is *solenoidal*, and ω_2 *irrotational*. $\mathrm{Pot}\,\omega_1$ and $\mathrm{Pot}\,\omega_2$, like $\mathrm{Pot}\,\omega$, will have in general definite values.

It is worth while to notice that there is only one way in which a vector function of position in space having a definite potential can be thus divided into solenoidal and irrotational parts having definite potentials. For if $\omega_1 + \epsilon$, $\omega_2 - \epsilon$ are two other such parts,

$$\nabla.\epsilon = 0 \quad \text{and} \quad \nabla \times \epsilon = 0.$$

Moreover, $\mathrm{Pot}\,\epsilon$ has in general a definite value, and therefore

$$\epsilon = \frac{1}{4\pi}\,\mathrm{Lap}\,\nabla \times \epsilon - \frac{1}{4\pi}\,\mathrm{New}\,\nabla.\epsilon = 0. \qquad \text{Q.E.D.}$$

99. To assist the memory of the student, some of the principal results of Nos. 93-98 may be expressed as follows:

Let ω_1 be any solenoidal vector function of position in space, ω_2 any irrotational vector function, and u any scalar function, satisfying the conditions that their potentials have in general definite values.

With respect to the solenoidal function ω_1, $\frac{1}{4\pi}\,\mathrm{Lap}$ and $\nabla\times$ are inverse operators; i.e.,

$$\frac{1}{4\pi}\,\mathrm{Lap}\,\nabla \times \omega_1 = \nabla \times \frac{1}{4\pi}\,\mathrm{Lap}\,\omega_1 = \omega_1.$$

Applied to the irrotational function ω_2, either of these operators gives zero; i.e.,

$$\mathrm{Lap}\,\omega_2 = 0, \qquad \nabla \times \omega_2 = 0.$$

With respect to the irrotational function ω_2, or the scalar function u, $\frac{1}{4\pi}\,\mathrm{New}$ and $-\nabla.$ are inverse operators; i.e.,

$$-\frac{1}{4\pi}\,\mathrm{New}\,\nabla.\omega_2 = \omega_2, \qquad -\nabla.\frac{1}{4\pi}\,\mathrm{New}\,u = u.$$

Applied to the solenoidal function ω_1, the operator $\nabla.$ gives zero; i.e.

$$\nabla.\omega_1 = 0.$$

Since the most general form of a vector function having in general a definite potential may be written $\omega_1 + \omega_2$, the effect of these operators on such a function needs no especial mention.

With respect to the solenoidal function ω_1, $\frac{1}{4\pi}\,\mathrm{Pot}$ and $\nabla \times \nabla\times$ are inverse operators; i.e.,

$$\frac{1}{4\pi}\,\mathrm{Pot}\,\nabla \times \nabla \times \omega_1 = \nabla \times \frac{1}{4\pi}\,\mathrm{Pot}\,\nabla \times \omega_1 = \nabla \times \nabla \times \frac{1}{4\pi}\,\mathrm{Pot}\,\omega_1 = \omega_1.$$

With respect to the irrotational function ω_2, $\frac{1}{4\pi}\,\mathrm{Pot}$ and $-\nabla\nabla.$ are inverse operators; i.e.,

$$-\frac{1}{4\pi}\,\mathrm{Pot}\,\nabla\nabla.\omega_2 = -\nabla\frac{1}{4\pi}\,\mathrm{Pot}\,\nabla.\omega_2 = -\nabla\nabla.\frac{1}{4\pi}\,\mathrm{Pot}\,\omega_2 = \omega_2.$$

With respect to any scalar or vector function having in general a definite potential $\frac{1}{4\pi}$ Pot and $-\nabla.\nabla$ are inverse operators; i.e.,

$$-\frac{1}{4\pi}\operatorname{Pot}\nabla.\nabla u = -\nabla.\frac{1}{4\pi}\operatorname{Pot}\nabla u = -\nabla.\nabla\frac{1}{4\pi}\operatorname{Pot}u = u,$$

$$-\frac{1}{4\pi}\operatorname{Pot}\nabla.\nabla[\omega_1+\omega_2] = -\nabla.\nabla\frac{1}{4\pi}\operatorname{Pot}[\omega_1+\omega_2] = \omega_1+\omega_2.$$

With respect to the solenoidal function ω_1, $-\nabla.\nabla$ and $\nabla\times\nabla\times$ are equivalent; with respect to the irrotational function ω_2, $\nabla.\nabla$ and $\nabla\nabla$ are equivalent; i.e.,

$$-\nabla.\nabla\omega_1 = \nabla\times\nabla\times\omega_1, \qquad \nabla.\nabla\omega_2 = \nabla\nabla.\omega_2.$$

100. *On the interpretation of the preceding formulæ*.—Infinite values of the quantity which occurs in a volume-integral as the coefficient of the element of volume will not necessarily make the value of the integral infinite, when they are confined to certain surfaces, lines, or points. Yet these surfaces, lines, or points may contribute a certain finite amount to the value of the volume-integral, which must be separately calculated, and in the case of surfaces or lines is naturally expressed as a surface- or line-integral. Such cases are easily treated by substituting for the surface, line, or point, a very thin shell, or filament, or a solid very small in all dimensions, within which the function may be supposed to have a very large value.

The only cases which we shall here consider in detail are those of surfaces at which the functions of position (u or ω) are discontinuous, and the values of ∇u, $\nabla\times\omega$, $\nabla.\omega$ thus become infinite. Let the function u have the value u_1 on the side of the surface which we regard as the negative, and the value u_2 on the positive side. Let $\Delta u = u_2 - u_1$. If we substitute for the surface a shell of very small thickness a, within which the value of u varies uniformly as we pass through the shell, we shall have $\nabla u = \nu\frac{\nabla u}{a}$ within the shell, ν denoting a unit normal on the positive side of the surface. The elements of volume which compose the shell may be expressed by $a[d\sigma]_0$, where $[d\sigma]_0$ is the magnitude of an element of the surface, $d\sigma$ being the vector element. Hence,

$$\nabla u\,dv = \nu\Delta u[d\sigma]_0 = \Delta u\,d\sigma.$$

Hence, when there are surfaces at which the values of u are discontinuous, the full value of Pot ∇u should always be understood as including the surface-integral

$$\int\!\!\int \frac{\Delta u'}{[\rho'-\rho]_0}\,d\sigma'$$

relating to such surfaces. ($\Delta u'$ and $d\sigma'$ are accented in the formula to indicate that they relate to the point ρ'.)

In the case of a vector function which is discontinuous at a surface, the expressions $\nabla.\omega\,dv$ and $\nabla\times\omega\,dv$, relating to the element of the shell which we substitute for the surface of discontinuity, are easily transformed by the principle that these expressions are the direct and skew surface-integrals of ω for the element of the shell. (See Nos. 55, 56.) The part of the surface-integrals relating to the edge of the element may evidently be neglected, and we shall have

$$\nabla.\omega\, dv = \omega_2.d\sigma - \omega_1.d\sigma = \Delta\omega.d\sigma,$$
$$\nabla \times \omega\, dv = d\sigma \times \omega_2 - d\sigma \times \omega_1 = d\sigma \times \Delta\omega.$$

Whenever, therefore, ω is discontinuous at surfaces, the expressions $\operatorname{Pot} \nabla.\omega$ and $\operatorname{New} \nabla.\omega$ must be regarded as implicitly including the surface-integrals

$$\int\int \frac{1}{[\rho' - \rho]_0} \Delta\omega'.d\sigma' \qquad \text{and} \qquad \int\int \frac{\rho' - \rho}{[\rho' - \rho]_0^3} \Delta\omega'.d\sigma'$$

respectively, relating to such surfaces, and the expressions $\operatorname{Pot} \nabla \times \omega$ and $\operatorname{Lap} \nabla \times \omega$ as including the surface-integrals

$$\int\int \frac{1}{[\rho' - \rho]_0} d\sigma' \times \Delta\omega' \qquad \text{and} \qquad \int\int \frac{\rho' - \rho}{[\rho' - \rho]_0^3} \times [d\sigma' \times \Delta\omega']$$

respectively, relating to such surfaces.

101. We have already seen that if ω is the curl of any vector function of position, $\nabla.\omega = 0$. (No. 68.) The converse is evidently true, whenever the equation $\nabla.\omega = 0$ holds throughout all space, and ω has in general a definite potential; for then

$$\omega = \nabla \times \frac{1}{4\pi} \operatorname{Lap} \omega.$$

Again, if $\nabla.\omega = 0$ within any aperiphractic space A, contained within finite boundaries, we may suppose that space to be enclosed by a shell B having its inner surface coincident with the surface of A. We may imagine a function of position ω', such that $\omega' = \omega$ in A, $\omega' = 0$ outside of the shell B, and the integral $\int\int\int \omega'.\omega'\, dv$ for B has the least value consistent with the conditions that the normal component of ω' at the outer surface is zero, and at the inner surface is equal to that of ω, and that in the shell $\nabla.\omega' = 0$ (compare No. 90). Then $\nabla.\omega' = 0$ throughout all space, and the potential of ω' will have in general a definite value. Hence,

$$\omega' = \nabla \times \frac{1}{4\pi} \operatorname{Lap} \omega',$$

and ω will have the same value within the space A.

102.[5] *Def.*—If ω is a vector function of position in space, the *Maxwellian*[6] of ω is a scalar function of position defined by the equation

$$\operatorname{Max} \omega = \int\int\int \frac{\rho' - \rho}{[\rho' - \rho]_0^3} .\omega'\, dv'.$$

[5] [The foregoing portion of this paper was printed in 1881, the rest in 1884.]

[6] The frequent occurrence of the integral in Maxwell's *Treatise on Electricity and Magnetism* has suggested this name.

(Compare No. 92.) From this definition the following properties are easily derived. It is supposed that the functions ω and u are such that their potentials have in general definite values.

$$\begin{aligned}\operatorname{Max}\omega &= \nabla .\operatorname{Pot}\omega = \operatorname{Pot}\nabla .\omega,\\ \nabla \operatorname{Max}\omega &= \nabla\nabla .\operatorname{Pot}\omega = \operatorname{New}\nabla .\omega,\\ \operatorname{Max}\nabla u &= -4\pi u,\\ 4\pi\omega &= \nabla\times\operatorname{Lap}\omega - \nabla\operatorname{Max}\omega.\end{aligned}$$

If the values of $\operatorname{Lap}\operatorname{Lap}\omega$, $\operatorname{New}\operatorname{Max}\omega$, and $\operatorname{Max}\operatorname{New}\ u$ are in general definite, we may add

$$\begin{aligned}4\pi\operatorname{Pot}\omega &= \operatorname{Lap}\operatorname{Lap}\omega - \operatorname{New}\operatorname{Max}\omega,\\ 4\pi\operatorname{Pot}u &= -\operatorname{Max}\operatorname{New}u.\end{aligned}$$

In other words: The Maxwellian is the divergence of the potential, $-\frac{\operatorname{Max}}{4\pi}$ and ∇ are inverse operators for scalars and irrotational vectors, for vectors in general $-\frac{1}{4\pi}\nabla\operatorname{Max}$ is an operator which separates the irrotational from the solenoidal part. For scalars and irrotational vectors, $\frac{-1}{4\pi}\operatorname{Max}\operatorname{New}$ and $\frac{-1}{4\pi}\operatorname{New}\operatorname{Max}$ give the potential, for solenoidal vectors $\frac{1}{4\pi}\operatorname{Lap}\operatorname{Lap}$ gives the potential, for vectors in general $\frac{-1}{4\pi}\operatorname{New}\operatorname{Max}$ gives the potential of the irrotational part, and $\frac{1}{4\pi}\operatorname{Lap}\operatorname{Lap}$ the potential of the solenoidal part.

103. *Def.*—The following double volume-integrals are of frequent occurrence in physical problems. They are all scalar quantities, and none of them functions of position in space, as are the single volume-integrals which we have been considering. The integrations extend over all space, or as far as the expression to be integrated has values other than zero.

The *mutual potential*, or *potential product*, of two scalar functions of position in space is defined by the equation

$$\begin{aligned}\operatorname{Pot}(u,w) &= \int\int\int\int\int\int \frac{uw'}{r}\,dv\,dv'\\ &= \int\int\int u\operatorname{Pot}w\,dv\\ &= \int\int\int w\operatorname{Pot}u\,dv.\end{aligned}$$

In the double volume-integral, r is the distance between the two elements of volume, and u relates to dv as w' to dv'.

The *mutual potential*, or *potential product*, of two vector functions of position in space is defined by the equation

$$\begin{aligned}\operatorname{Pot}(\phi,\omega) &= \int\int\int\int\int\int \frac{\phi .\omega'}{r}\,dv\,dv'\\ &= \int\int\int \phi .\operatorname{Pot}\omega\,dv\\ &= \int\int\int \omega .\operatorname{Pot}\phi\,dv.\end{aligned}$$

The *mutual Laplacian*, or *Laplacian product*, of two vector functions of position in space is defined by the equation

$$\begin{aligned}\mathrm{Lap}(\phi,\omega) &= \int\int\int\int\int\int \omega.\frac{\rho'-\rho}{r^3}\times\phi'\,dv\,dv' \\ &= \int\int\int \omega.\,\mathrm{Lap}\,\phi\,dv \\ &= \int\int\int \phi.\,\mathrm{Lap}\,\omega\,dv.\end{aligned}$$

The *Newtonian product* of a scalar and a vector function of position in space is defined by the equation

$$\mathrm{New}(u,\omega) = \int\int\int\int\int\int \omega.\frac{\rho'-\rho}{r^3}u'\,dv\,dv' = \int\int\int \omega.\,\mathrm{New}\,u\,dv.$$

The *Maxwellian product* of a vector and a scalar function of position in space is defined by the equation

$$\begin{aligned}\mathrm{Max}(\omega,u) &= \int\int\int\int\int\int u\frac{\rho'-\rho}{r^3}.\omega'\,dv\,dv' \\ &= \int\int\int u\,\mathrm{Max}\,\omega\,dv \\ &= -\,\mathrm{New}(u,\omega).\end{aligned}$$

It is of course supposed that u, w, ϕ, ω are such functions of position that the above expressions have definite values.

104. By No. 97,

$$4\pi u\,\mathrm{Pot}\,w = -\nabla.\,\mathrm{New}\,u\,\mathrm{Pot}\,w = -\nabla.[\mathrm{New}\,u\,\mathrm{Pot}\,w] + \mathrm{New}\,u.\,\mathrm{New}\,w.$$

The volume-integral of this equation gives

$$4\pi\,\mathrm{Pot}(u,w) = \int\int\int \mathrm{New}\,u.\,\mathrm{New}\,w\,dv,$$

if the integral

$$\int\int d\sigma.\,\mathrm{New}\,u\,\mathrm{Pot}\,w,$$

for a closed surface, vanishes when the space included by the surface is indefinitely extended in all directions. This will be the case when everywhere outside of certain assignable limits the values of u and w are zero.

Again, by No. 102,

$$\begin{aligned}4\pi\omega.\,\mathrm{Pot}\,\phi &= \nabla\times\mathrm{Lap}\,\omega.\,\mathrm{Pot}\,\phi - \nabla\,\mathrm{Max}\,\omega.\,\mathrm{Pot}\,\phi \\ &= \nabla.[\mathrm{Lap}\,\omega\times\mathrm{Pot}\,\phi] + \mathrm{Lap}\,\omega.\,\mathrm{Lap}\,\phi \\ &\quad - \nabla.[\mathrm{Max}\,\omega\,\mathrm{Pot}\,\phi] + \mathrm{Max}\,\omega\,\mathrm{Max}\,\phi.\end{aligned}$$

The volume-integral of this equation gives

$$4\pi \,\mathrm{Pot}(\phi, \omega) = \int\int\int \mathrm{Lap}\,\phi.\,\mathrm{Lap}\,\omega\, dv + \int\int\int \mathrm{Max}\,\phi\,\mathrm{Max}\,\omega\, dv,$$

if the integrals

$$\int\int d\sigma.\,\mathrm{Lap}\,\omega \times \mathrm{Pot}\,\phi, \qquad \int\int d\sigma.\,\mathrm{Pot}\,\phi\,\mathrm{Max}\,\omega,$$

for a closed surface vanish when the space included by the surface is indefinitely extended in all directions. This will be the case if everywhere outside of certain assignable limits the values of ϕ and ω are zero.

Chapter 3

CONCERNING LINEAR VECTOR FUNCTIONS.

105. *Def.*—A vector function of a vector is said to be *linear*, when the function of the sum of any two vectors is equal to the sum of the functions of the vectors. That is, if

$$\operatorname{func}.[\rho + \rho'] = \operatorname{func}.[\rho] + \operatorname{func}.[\rho']$$

for all values of ρ and ρ', the function is linear. In such cases it is easily shown that

$$\operatorname{func}.[a\rho + b\rho' + c\rho'' + \operatorname{etc}.] = a\operatorname{func}.[\rho] + b\operatorname{func}.[\rho'] + c\operatorname{func}.[\rho''] + \operatorname{etc}.$$

106. An expression of the form

$$\alpha\lambda.\rho + \beta\mu.\rho + \operatorname{etc}.$$

evidently represents a linear function of ρ and may be conveniently written in the form

$$\{\alpha\lambda + \beta\mu + \operatorname{etc}.\}.\rho.$$

The expression

$$\rho.\alpha\lambda + \rho.\beta\mu + \operatorname{etc}.,$$

or

$$\rho.\{\alpha\lambda + \beta\mu + \operatorname{etc}.\},$$

also represents a linear function of ρ which is, in general, different from the preceding, and will be called its *conjugate*.

107. *Def.*—An expression of the form $\alpha\lambda$ or $\beta\mu$ will be called a *dyad*. An expression consisting of any number of dyads united by the signs $+$ or $-$ will be called a *dyadic binomial, trinomial*, etc., as the case may be, or more briefly, a *dyadic*. The latter term will be used so as to include the case of a single dyad. When we desire to express a dyadic by a single letter, the Greek capitals will be used, except such as are like the Roman, and also Δ and Σ. The letter I will also be used to represent a certain dyadic, to be mentioned hereafter.

Since any linear vector function may be expressed by means of a dyadic (as we shall see more particularly hereafter, see No. 110), the study of such functions, which is evidently of primary importance in the theory of vectors, may be reduced to that of dyadics.

108. *Def.*—Any two dyadics Φ and Ψ are equal,

$$\begin{array}{rrcll} \text{when} & \Phi.\rho & = & \Psi.\rho & \text{for all values of } \rho, \\ \text{or, when} & \rho.\Phi & = & \rho.\Psi & \text{for all values of } \rho, \\ \text{or, when} & \sigma.\Phi.\rho & = & \sigma.\Psi.\rho & \text{for all values of } \sigma \text{ and of } \rho. \end{array}$$

The third condition is easily shown to be equivalent both to the first and to the second. The three conditions are therefore equivalent.

It follows that $\Phi = \Psi$, if $\Phi.\rho = \Psi.\rho$, or $\phi.\Phi = \rho.\Psi$, for three non- complanar values of ρ.

109. *Def.*—We shall call the vector $\Phi.\rho$ the (direct) product of Φ and ρ, the vector $\rho.\Phi$ the (direct) product of ρ and Φ, and the scalar $\rho.\Phi.\rho$ the (direct) product of σ, Φ, and ρ.

In the combination $\Phi.\rho$, we shall say that Φ is used as a *prefactor*, in the combination $\rho.\Phi$, as a *postfactor*.

110. If τ is any linear function of ρ, and for $\rho = i$, $\rho = j$, $\rho = k$, the values of τ are respectively α, β, and γ, we may set

$$\tau = \{\alpha i + \beta j + \gamma k\}.\rho,$$

and also
$$\tau = \rho.\{i\alpha + j\beta + k\gamma\}.$$

Therefore, any linear function may be expressed by a dyadic as prefactor and also by a dyadic as postfactor.

111. *Def.*—We shall say that a dyadic is multiplied by a scalar when one of the vectors of each of its component dyads is multiplied by that scalar. It is evidently immaterial to which vector of any dyad the scalar factor is applied. The product of the dyadic Φ and the scalar a may be written either $a\Phi$ or Φa. The minus sign before a dyadic reverses the signs of all its terms.

112. The sign $+$ in a dyadic, or connecting dyadics, may be regarded as expressing addition, since the combination of dyads and dyadics with this sign is subject to the laws of association and commutation.

113. The combination of vectors in a dyad is evidently distributive. That is,

$$[\alpha + \beta + \text{etc}.][\lambda + \mu + \text{etc}.] = \alpha\lambda + \alpha\mu + \beta\lambda + \beta\mu + \text{etc}.$$

We may therefore regard the dyad as a kind of product of the two vectors of which it is formed. Since this kind of product is not commutative, we shall have occasion to distinguish the factors as *antecedent* and *consequent*.

114. Since any vector may be expressed as a sum of i, j, and k with scalar coefficients, every dyadic may be reduced to a sum of the nine dyads

$$ii, ij, ik, ji, jj, jk, ki, kj, kk,$$

with scalar coefficients. Two such sums cannot be equal according to the definitions of No. 108, unless their coefficients are equal each to each. Hence dyadics are equal only when their equality can be deduced from the principle that the operation of forming a dyad is a distributive one.

On this account, we way regard the dyad as the most general form of product of two vectors. We shall call it the indeterminate product. The complete determination of a single dyad involves five independent scalars, of a dyadic, nine.

115. It follows from the principles of the last paragraph that if

$$\Sigma\, \alpha\beta = \Sigma\, \kappa\lambda,$$

then
$$\Sigma\, \alpha \times \beta = \Sigma\, \kappa \times \lambda,$$

and
$$\Sigma\, \alpha.\beta = \Sigma\, \kappa.\lambda.$$

In other words, the vector and the scalar obtained from a dyadic by insertion of the sign of skew or direct multiplication in each dyad are both independent of the particular form in which the dyadic is expressed.

We shall write $\Phi_\times$ and Φ_S to indicate the vector and the scalar thus obtained.

$$\Phi_\times = (j.\Phi.k - k.\Phi.j)i + (k.\Phi.i - i.\Phi.k)j + (i.\Phi.j - j.\Phi.i)k,$$

$$\Phi_S = i.\Phi.i + j.\Phi.j + k.\Phi.k,$$

as is at once evident, if we suppose Φ to be expanded in terms of ii, ij, etc.

116. *Def.*—The (*direct*) *product* of two dyads (indicated by a dot) is the dyad formed of the first and last of the four factors, multiplied by the direct product of the second and third. That is,

$$\{\alpha\beta\}.\{\gamma\delta\} = \alpha\beta.\gamma\delta = \beta.\gamma\, \alpha\delta.$$

The (direct) product of two dyadics is the sum of all the products formed by prefixing a term of the first dyadic to a term of the second. Since the direct product of one dyadic with another is a dyadic, it may be multiplied in the same way by a third, and so on indefinitely. This kind of multiplication is evidently associative, as well as distributive. The same is true of the direct product of a series of factors of which the first and the last are either dyadics or vectors, and the other factors are dyadics. Thus the values of the expressions

$$\alpha.\Phi.\Theta.\Psi.\beta, \qquad \alpha.\Phi.\Theta, \qquad \Phi.\Theta.\Psi.\beta, \qquad \Phi.\Theta.\Psi$$

will not be affected by any insertion of parentheses. But this kind of multiplication is not commutative, except in the case of the direct product of two vectors.

117. *Def.*—The expressions $\Phi \times \rho$ and $\rho \times \Phi$ represent dyadics which we shall call the *skew* products of Φ and ρ. If

$$\Phi = \alpha\lambda + \beta\mu + \mathrm{etc}\,.$$

these skew products are defined by the equations

$$\Phi \times \rho = \alpha\lambda \times \rho + \beta\mu \times \rho + \text{etc}.,$$
$$\rho \times \Phi = \rho \times \alpha\lambda + \rho \times \beta\mu + \text{etc}.$$

It is evident that

$$\{\rho \times \Phi\}.\Psi = \rho \times \{\Phi.\Psi\}, \qquad \Psi.\{\Phi \times \rho\} = \{\Psi.\Phi\} \times \rho,$$

$$\{\rho \times \Phi\}.\alpha = \rho \times \{\Phi.\alpha\}, \qquad \alpha.\{\Phi \times \rho\} = [\alpha.\Phi] \times \rho,$$
$$\{\rho \times \Phi\} \times \alpha = \rho \times \{\Phi \times \alpha\}.$$

We may therefore write without ambiguity

$$\rho \times \Phi.\Psi, \quad \Psi.\Phi \times \rho, \quad \rho \times \Phi.\alpha, \quad \alpha.\Phi \times \rho, \quad \rho \times \Phi.\alpha.$$

This may be expressed a little more generally by saying that the associative principle enunciated in No. 116 may be extended to cases in which the initial or final vectors are connected with the other factors by the sign of skew multiplication.

Moreover,

$$\alpha.\rho \times \Phi = [\alpha \times \rho].\Phi \quad \text{and} \quad \Phi \times \rho.\alpha = \Phi.[\rho \times \alpha].$$

These expressions evidently represent vectors. So

$$\Psi.\{\rho \times \Phi\} = \{\Psi \times \rho\}.\Phi.$$

These expressions represent dyadics. The braces cannot be omitted without ambiguity.

118. Since all the antecedents or all the consequents in any dyadic may be expressed in parts of any three non-complanar vectors, and since the sum of any number of dyads having the same antecedent or the same consequent may be expressed by a single dyad, it follows that any dyadic may be expressed as the sum of three dyads, and so, that either the antecedents or the consequents shall be any desired non-complanar vectors, but only in one way when either the antecedents or the consequents are thus given.

In particular, the dyadic

$$aii + bij + cik + a'ji + b'jj + c'jk + a''ki + b''kj + c''kk,$$

which may for brevity be written

$$\left\{ \begin{array}{ccc} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{array} \right\}$$

is equal to $$\alpha i + \beta j + \gamma k,$$

where

$$\alpha = ai + a'j + a''k,$$
$$\beta = bi + b'j + b''k,$$
$$\gamma = ci + c'j + c''k,$$

and to $$i\lambda + j\mu + k\nu,$$

where

$$\lambda = ai + bj + ck$$
$$\mu = a'i + b'j + c'k$$
$$\nu = a''i + b''j + c''k.$$

119. By a similar process, the sum of three dyads may be reduced to the sum of two dyads, whenever either the antecedents or the consequents are complanar, and only in such cases. To prove the latter point, let us suppose that in the dyadic

$$\alpha\lambda + \beta\mu + \gamma\nu$$

neither the antecedents nor the consequents are complanar. The vector

$$\{\alpha\lambda + \beta\mu + \gamma\nu\}.\rho$$

is a linear function of ρ which will be parallel to α when ρ is perpendicular to μ and ν, which will be parallel to β when ρ is perpendicular to ν and λ, and which will be parallel to γ when ρ is perpendicular to λ and μ. Hence, the function may be given any value whatever by giving the proper value to ρ. This would evidently not be the case with the sum of two dyads. Hence, by No. 108, this dyadic cannot be equal to the sum of two dyads.

120. In like manner, the sum of two dyads may be reduced to a single dyad, if either the antecedents or the consequents are parallel, and only in such cases.

A sum of three dyads cannot be reduced to a single dyad, unless either their antecedents or consequents are parallel, or both antecedents and consequents are (separately) complanar. In the first case the reduction can always be made, in the second, occasionally.

121. *Def.*—A dyadic which cannot be reduced to the sum of less than three dyads will be called *complete*.

A dyadic which can be reduced to the sum of two dyads will be called *planar*. When the plane of the antecedents coincides with that of the consequents, the dyadic will be called *uniplanar*. These planes are invariable for a given dyadic, although the dyadic may be so expressed that either the two antecedents or the two consequents may have any desired values (which are not parallel) within their planes.

A dyadic which can be reduced to a single dyad will be called *linear*. When the antecedent and consequent are parallel, it will be called *unilinear*.

A dyadic is said to have the value zero when all its terms vanish.

122. If we set

$$\sigma = \Phi.\rho, \qquad \tau = \rho.\Phi,$$

and give ρ all possible values, σ and τ will receive all possible values, if Φ is complete. The values of σ and τ will be confined each to a plane if Φ is planar, which planes will coincide if Φ is uniplanar. The values of σ and τ will be confined each to a line if Φ is linear, which lines will coincide if Φ is unilinear.

123. The products of complete dyadics are complete, of complete and planar dyadics are planar, of complete and linear dyadics are linear.

The products of planar dyadics are planar, except that when the plane of the consequents of the first dyadic is perpendicular to the plane of the antecedents of the second dyadic, the product reduces to a linear dyadic.

The products of linear dyadics are linear, except that when the consequent of the first is perpendicular to the antecedent of the second, the product reduces to zero.

The products of planar and linear dyadics are linear, except when, the planar preceding, the plane of its consequents is perpendicular to the antecedent of the linear, or, the linear preceding, its consequent is perpendicular to the plane of the antecedents of the planar. In these cases the product is zero.

All these cases are readily proved, if we set

$$\sigma = \Phi.\Psi.\rho,$$

and consider the limits within which σ varies, when we give ρ all possible values.

The products $\Psi \times \rho$ and $\rho \times \Phi$ are evidently planar dyadics.

124. *Def.*—A dyadic Φ is said to be an *idemfactor*, when

$$\Phi.\rho = \rho \quad \text{for all values of } \rho,$$

or when

$$\rho.\Phi = \rho \quad \text{for all values of } \rho.$$

If either of these conditions holds true, Φ must be reducible to the form

$$ii + jj + kk.$$

Therefore, both conditions will hold, if either does. All such dyadics are equal, by No. 108. They will be represented by the letter I.

The direct product of an idemfactor with another dyadic is equal to that dyadic. That is,

$$I.\Phi = \Phi, \qquad \Phi.I = \Phi,$$

where Φ is any dyadic.

A dyadic of the form

$$\alpha\alpha' + \beta\beta' + \gamma\gamma',$$

in which α', β', γ' are the reciprocals of α, β, γ, is an idemfactor. (See No. 38.) A dyadic trinomial cannot be an idemfactor, unless its antecedents and consequents are reciprocals.

125. If one of the direct products of two dyadics is an idemfactor, the other is also. For, if $\Phi.\Psi = I$,

$$\sigma.\Phi.\Psi = \sigma$$

for all values of σ, and Φ is complete;

$$\sigma.\Phi.\Psi.\Phi = \sigma.\Phi$$

for all values of σ, therefore for all values of $\sigma.\Phi$, and therefore $\Psi.\Phi = I$.

Def.—In this case, either dyadic is called the *reciprocal* of the other.

It is evident that an incomplete dyadic cannot have any (finite) reciprocal.

Reciprocals of the same dyadic are equal. For if Φ and Ψ are both reciprocals of Ω,

$$\Phi = \Phi.\Omega.\Psi = \Psi.$$

If two dyadics are reciprocals, the operators formed by using these dyadics as prefactors are inverse, also the operators formed by using them as postfactors.

126. The reciprocal of any complete dyadic

$$\alpha\lambda + \beta\mu + \gamma\nu$$

is
$$\lambda'\alpha' + \mu'\beta' + \nu'\gamma',$$

where α', β', γ' are the reciprocals of α, β, γ, and λ', μ', ν' are the reciprocals of λ, μ, ν. (See No. 38.)

127. *Def.*—We shall write Φ^{-1} for the reciprocal of any (complete) dyadic Φ also Φ^2 for $\Phi.\Phi$, etc., and Φ^{-2}, for $\Phi^{-1}.\Phi^{-1}$, etc. It is evident that Φ^{-n} is the reciprocal of Φ^n.

128. In the reduction of equations, if we have

$$\Phi.\Psi = \Phi.\Omega,$$

we may cancel the Φ (which is equivalent to multiplying by Φ^{-1}) if Φ is a complete dyadic, but not otherwise. The case is the same with such equations as

$$\Phi.\sigma = \Phi.\rho, \quad \Psi.\Phi = \Omega.\Phi, \quad \rho.\Phi = \sigma.\Phi.$$

To cancel an incomplete dyadic in such cases would be analogous to cancelling a zero factor in algebra.

129. *Def.*—If in any dyadic we transpose the factors in each term, the dyadic thus formed is said to be *conjugate* to the first. Thus

$$\alpha\lambda + \beta\mu + \gamma\nu \quad \text{and} \quad \lambda\alpha + \mu\beta + \nu\lambda$$

are conjugate to each other. A dyadic of which the value is not altered by such transposition is said to be *self-conjugate*. The conjugate of any dyadic Φ may be written Φ_C. It is evident that

$$\rho.\Phi = \Phi_C.\rho \quad \text{and} \quad \Phi.\rho = \rho.\Phi_C.$$

$\Phi_C.\rho$ and $\Phi.\rho$ are conjugate functions of ρ (See No. 106.) Since $\{\Phi_C\}^2 = \{\Phi^2\}_C$, we may write Φ^2_C, etc., without ambiguity.

130. The reciprocal of the product of any number of dyadics is equal to the product of their reciprocals taken in inverse order. Thus

$$\{\Phi.\Psi.\Omega\}^{-1} = \Omega^{-1}.\Psi^{-1}.\Phi^{-1}.$$

The conjugate of the product of any number of dyadics is equal to the product of their conjugates taken in inverse order. Thus

$$\{\Phi.\Psi.\Omega\}_C = \Omega_C.\Psi_C.\Phi_C.$$

Hence, since

$$\Phi_C.\{\Phi^{-1}\}_C = \{\Phi^{-1}.\Phi\}_C = I,$$
$$\{\Phi^{-1}\}_C = \{\Phi_C\}^{-1},$$

and we may write Φ_C^{-1} without ambiguity.

131. It is sometimes convenient to be able to express by a dyadic taken in direct multiplication the same operation which would be effected by a given vector (α) in skew multiplication. The dyadic $I \times \alpha$ will answer this purpose. For, by No. 117,

$$\{I \times \alpha\}.\rho = \alpha \times \rho, \qquad \rho.\{I \times \alpha\} = \rho \times \alpha,$$
$$\{I \times \alpha\}.\Phi = \alpha \times \Phi, \qquad \Phi.\{I \times \alpha\} = \Phi \times \alpha.$$

The same is true of the dyadic $\alpha \times I$, which is indeed identical with $I \times \alpha$, as appears from the equation $I.\{\alpha \times I\} = \{I \times \alpha\}.I$.

If α is a unit vector,

$$\{I \times \alpha\}^2 = -\{I - \alpha\alpha\},$$
$$\{I \times \alpha\}^3 = -I \times \alpha,$$
$$\{I \times \alpha\}^4 = I - \alpha\alpha,$$
$$\{I \times \alpha\}^5 = I \times \alpha,$$
$$\text{etc}\,.$$

If i, j, k are a normal system of unit vectors

$$I \times i = i \times I = kj - jk.$$
$$I \times j = j \times I = ik - ki,$$
$$I \times k = k \times I = ji - ij.$$

If α and β are any vectors,

$$[\alpha \times \beta] \times I = I \times [\alpha \times \beta] = \beta\alpha - \alpha\beta.$$

That is, the vector $\alpha \times \beta$ as a pre- or post-factor in skew multiplication is equivalent to the dyadic $\{\beta\alpha - \alpha\beta\}$ taken as pre- or post-factor in direct multiplication.

$$[\alpha \times \beta] \times \rho = \{\beta\alpha - \alpha\beta\}.\rho,$$
$$\rho \times [\alpha \times \beta] = \rho\{\beta\alpha - \alpha\beta\}.$$

This is essentially the theorem of No. 27, expressed in a form more symmetrical and more easily remembered.

132. The equation

$$\alpha\,\beta \times \gamma + \beta\,\gamma \times \alpha + \gamma\,\alpha \times \beta = \alpha.\beta \times \gamma I.$$

gives, on multiplication by any vector ρ the identical equation

$$\rho.\alpha\,\beta \times \gamma + \rho.\beta\,\gamma \times \alpha + \rho.\gamma\,\alpha \times \beta = \alpha.\beta \times \gamma\rho.$$

(See No. 37.) The former equation is therefore identically true. (See No. 108.) It is a little more general than the equation

$$\alpha\alpha' + \beta\beta' + \gamma\gamma' = I,$$

which we have already considered (No. 124), since, in the form here given, it is not necessary that α, β, and γ should be non-complanar. We may also write

$$\beta \times \gamma\alpha + \gamma \times \alpha\beta + \alpha \times \beta\gamma = \alpha.\beta \times \gamma I.$$

Multiplying this equation by ρ as prefactor (or the first equation by ρ as postfactor), we obtain

$$\rho.\beta \times \gamma\alpha + \rho.\gamma \times \alpha\beta + \rho.\alpha \times \beta\gamma = \alpha.\beta \times \gamma\rho.$$

(Compare No. 37.) For three complanar vectors we have

$$\alpha\beta \times \gamma + \beta\gamma \times \alpha + \gamma\alpha \times \beta = 0.$$

Multiplying this by ν, a unit normal to the plane of α, β, and γ we have

$$\alpha\beta \times \gamma.\nu + \beta\gamma \times \alpha.\nu + \gamma\alpha \times \beta.\nu = 0.$$

This equation expresses the well-known theorem that if the geometrical sum of three vectors is zero, the magnitude of each vector is proportional to the sine of the angle between the other two. It also indicates the numerical coefficients by which one of three complanar vectors may be expressed in parts of the other two.

133. *Def.*—If two dyadics Φ and Ψ are such that

$$\Phi.\Psi = \Psi.\Phi,$$

they are said to be *homologous*.

If any number of dyadics are homologous to one another, and any other dyadics are formed from them by the operations of taking multiples, sums, differences, powers, reciprocals, or products, such dyadics will be homologous to each other and to the original dyadics. This requires demonstration only in regard to reciprocals. Now if

$$\Phi.\Psi = \Psi.\Phi,$$
$$\Psi.\Phi^{-1} = \Phi^{-1}.\Phi.\Psi.\Phi^{-1} = \Phi^{-1}.\Psi.\Phi.\Phi^{-1} = \Phi^{-1}.\Psi.$$

That is, Φ^{-1} is homologous to Ψ, if Φ is.

134. If we call $\Psi.\Phi^{-1}$ or $\Phi^{-1}.\Psi$ the quotient of Ψ and Φ, we may say that the rules of addition, subtraction, multiplication and division of homologous dyadics are identical with those of arithmetic or ordinary algebra, except that limitations analogous to those respecting zero in algebra must be observed with respect to all incomplete dyadics.

It follows that the algebraic and higher analysis of homologous dyadics is substantially identical with that of scalars.

135. It is always possible to express a dyadic in three terms, so that both the antecedents and the consequents shall be perpendicular among themselves.

To show this for any dyadic Φ, let us set

$$\rho' = \Phi.\rho,$$

ρ being a unit-vector, and consider the different values of ρ' for all possible directions of ρ. Let the direction of the unit vector i be so determined that when ρ coincides with i, the value of ρ' shall be at least as great as for any other direction of ρ. And let the direction of the unit vector j be so determined that when ρ coincides with j, the value of ρ' shall be at least as great as for any other direction of ρ which is perpendicular to i. Let k have its usual position with respect to i and j. It is evidently possible to express Φ in the form

$$\alpha i + \beta j + \gamma k.$$

We have therefore

$$\rho' = \{\alpha i + \beta j + \gamma k\}.\rho,$$

and

$$d\rho' = \{\alpha i + \beta j + \gamma k\}.d\rho.$$

Now the supposed property of the direction of i requires that when ρ coincides with i and $d\rho$ is perpendicular to i, $d\rho'$ shall be perpendicular to ρ', which will then be parallel to α. But if $d\rho$ is parallel to j or k, it will be perpendicular to i, and $d\rho'$ will be parallel to β or γ, as the case may be. Therefore β and γ are perpendicular to α. In the same way it may be shown that the condition relative to j requires that γ shall be perpendicular to β. We may therefore set

$$\Phi = ai'i + bj'j + ck'k,$$

where i', j', k', like i, j, k, constitute a normal system of unit vectors (see No. 11), and a, b, c are scalars which may be either positive or negative.

It makes an important difference whether the number of these scalars which are negative is even or odd. If two are negative, say a and b, we may make them positive by reversing the directions of $i.$ and j'. The vectors i', j', k' will still constitute a normal system. But if we should reverse the directions of an odd number of these vectors, they would cease to constitute a normal system, and to be superposable upon the system i, j, k. We may, however, always set either

$$\Phi = ai'i + bj'j + ck'k,$$

or

$$\Phi = -\{ai'i + bj'j + ck'k\},$$

with positive values of a, b, and c. At the limit between these cases are the planar dyadics, in which one of the three terms vanishes, and the dyadic reduces to the form

$$ai'i + bj'j,$$

in which a and b may always be made positive by giving the proper directions to i' and j'.

If the numerical values of a, b, c are all unequal, there will be only one way in which the value of Φ may be thus expressed. If they are not all unequal, there will be an infinite number of ways in which Φ may be thus expressed, in all of which the three scalar coefficients will have the same values with exception of the changes of signs mentioned above. If the three values are numerically identical, we may give to either system of normal vectors an arbitrary position.

136. It follows that any self-conjugate dyadic may be expressed in the form

$$aii + bjj + ckk,$$

where i, j, k are a normal system of unit vectors, and a, b, c are positive or negative scalars.

137. Any dyadic may be divided into two parts, of which one shall be self-conjugate, and the other of the form $I \times \alpha$. These parts are found by taking half the sum and half the difference of the dyadic and its conjugate. It is evident that

$$\Phi = \frac{1}{2}\{\Phi + \Phi_C\} + \frac{1}{2}\{\Phi - \Phi_C\}.$$

Now $\frac{1}{2}\{\Phi + \Phi_C\}$ is self-conjugate, and

$$\frac{1}{2}\{\Phi - \Phi_C\} = I \times [-\frac{1}{2}\Phi_\times].$$

(See No. 131.)

Rotations and Strains.

138. To illustrate the use of dyadics as operators, let us suppose that a body receives such a displacement that

$$\rho' = \Phi.\rho,$$

ρ and ρ' being the position-vectors of the same point of the body in its initial and subsequent positions. The same relation will hold of the vectors which unite any two points of the body in their initial and subsequent positions. For if ρ_1, ρ_2 are the original position-vectors of the points, and ρ_1', ρ_2' their final position-vectors, we have

$$\rho_1' = \Phi.\rho_1, \qquad \rho_2' = \Phi.\rho_2,$$

whence

$$\rho_2' - \rho_1' = \Phi.[\rho_2 - \rho_1].$$

In the most general case, the body is said to receive a *homogeneous strain*. In special cases, the displacement reduces to a rotation. Lines in the body initially straight and parallel will be straight and parallel after the displacement, and surfaces initially plane and parallel will be plane and parallel after the displacement.

139. The vectors (σ, σ') which represent any plane surface in the body in its initial and final positions will be linear functions of each other. (This will appear, if we consider the four sides of a tetrahedron in the body.) To find the relation of the dyadics which express σ' as a function of σ, and ρ' as a function of ρ, let

$$\rho' = \{\alpha\lambda + \beta\mu + \gamma\nu\}.\rho,$$

Then, if we write λ', μ', ν' for the reciprocals of λ, μ, ν, the vectors λ', μ', ν' become by the strain α, β, γ. Therefore the surfaces $\mu' \times \nu'$, $\nu' \times \lambda'$, $\lambda' \times \mu'$ become $\beta \times \gamma$, $\gamma \times \alpha$, $\alpha \times \beta$. But $\mu' \times \nu'$, $\nu' \times \lambda'$, $\lambda' \times \mu'$ are the reciprocals of $\mu \times \nu$, $\nu \times \lambda$, $\lambda \times \mu$. The relation sought is therefore

$$\sigma' = \{\beta \times \gamma\, \mu \times \nu + \gamma \times \alpha\, \nu \times \lambda + \alpha \times \beta\, \lambda \times \mu\}.\sigma.$$

140. The volume $\lambda'.\mu' \times \nu'$ becomes by the strain $\alpha.\beta \times \gamma$. The unit of volume becomes therefore $(\alpha.\beta \times \gamma)(\lambda.\mu \times \nu)$.

Def.—It follows that the scalar product of the three antecedents multiplied by the scalar product of the three consequents of a dyadic expressed as a trinomial is independent of the particular form in which the dyadic is thus expressed. This quantity is the determinant of the coefficients of the nine terms of the form

$$aii + bij + \text{etc}.,$$

into which the dyadic may be expanded. We shall call it the *determinant* of the dyadic, and shall denote it by the notation

$$|\Phi|$$

when the dyadic is expressed by a single letter.

If a dyadic is incomplete, its determinant is zero, and conversely.

The determinant of the product of any number of dyadics is equal to the product of their determinants. The determinant of the reciprocal of a dyadic is the reciprocal of the determinant of that dyadic. The determinants of a dyadic and its conjugate are equal.

The relation of the surfaces σ' and σ may be expressed by the equation

$$\sigma' = |\Phi|\Phi_C^{-1}.\sigma.$$[1]

141. Let us now consider the different cases of rotation and strain as determined by the nature of the dyadic Φ.

If Φ is reducible to the form

$$i'i + j'j + k'k,$$

[1] [See note at end of this paper.]

i, j, k, i', j', k' being normal systems of unit vectors (see No. 11), the body will suffer no change of form. For if

$$\rho = xi + yj + zk,$$

we shall have

$$\rho' = xi' + yj' + zk'.$$

Conversely, if the body suffers no change of form, the operating dyadic is reducible to the above form. In such cases, it appears from simple geometrical considerations that the displacement of the body may be produced by a rotation about a certain axis. A dyadic reducible to the form

$$i'i + j'j + k'k$$

may therefore be called a *versor*.

142. The conjugate operator evidently produces the reverse rotation. A versor, therefore, is the reciprocal of its conjugate.

Conversely, if a dyadic is the reciprocal of its conjugate, it is either a versor, or a versor multiplied by -1. For the dyadic may be expressed in the form

$$\alpha i + \beta j + \gamma k.$$

Its conjugate will be

$$i\alpha + j\beta + k\gamma.$$

If these are reciprocals, we have

$$\{\alpha i + \beta j + \gamma k\}.\{i\alpha + j\beta + k\gamma\} = \alpha\alpha + \beta\beta + \gamma\gamma = I.$$

But this relation cannot subsist unless α, β, γ are reciprocals to themselves, i.e., unless they are mutually perpendicular unit-vectors. Therefore, they either are a normal system of unit-vectors, or will become such if their directions are reversed. Therefore, one of the dyadics

$$\alpha i + \beta j + \gamma k \quad \text{and} \quad -\alpha i - \beta j - \gamma k$$

is a versor.

The criterion of a versor way therefore be written

$$\Phi.\Phi_C = I, \quad \text{and} \quad |\Phi| = 1.$$

For the last equation we may substitute

$$|\Phi| > 0, \quad \text{or} \quad |\Phi| >< -1.$$

It is evident that the resultant of successive finite rotations is obtained by multiplication of the versors.

143. If we take the axis of the rotation for the direction of i, i' will have the same direction, and the versor reduces to the form

$$ii + j'j + k'k,$$

in which i, j, k and i, j', k' are normal systems of unit vectors.

We may set

$$j' = \cos q\, j + \sin q\, k,$$
$$k' = \cos q\, k - \sin q\, j,$$

and the versor reduces to

$$ii + \cos q\{jj + kk\} + \sin q\{kj - jk\},$$

or

$$ii + \cos q\{I - ii\} + \sin q\, I \times i,$$

where q is the angle of rotation, measured from j toward k, if the versor is used as a prefactor.

144. When any versor Φ is used as a prefactor, the vector $-\Phi_\times$ will be parallel to the axis of rotation, and equal in magnitude to twice the sine of the angle of rotation measured counter-clockwise as seen from the direction in which the vector points. (This will appear if we suppose Φ to be represented in the form given in the last paragraph.) The scalar Φ_S will be equal to unity increased by twice the cosine of the same angle. Together, $-\Phi_\times$ and Φ_S determine the versor without ambiguity. If we set

$$\theta = \frac{-\Phi_\times}{1 + \Phi_S},$$

the magnitude of θ will be

$$\frac{2\sin q}{2 + 2\cos q} \quad \text{or} \quad \tan\frac{1}{2}q,$$

where q is measured counter-clockwise as seen from the direction in which θ points. This vector θ, which we may call the *vector semitangent of version*, determines the versor without ambiguity.

145. The versor Φ may be expressed in terms of θ in various ways. Since Φ (as prefactor) changes $\alpha - \theta \times \alpha$ into $\alpha + \theta \times \alpha$ (α being any vector), we have

$$\Phi = \{I + I \times \theta\}.\{I - I \times \theta\}^{-1}.$$

Again

$$\Phi = \frac{\theta\theta + \{I + I \times \theta\}^2}{1 + \theta.\theta} = \frac{(1 - \theta.\theta)I + 2\theta\theta + 2I \times \theta}{1 + \theta.\theta},$$

as will be evident on considering separately in the expression $\Phi.\rho$ the components perpendicular and parallel to θ, or on substituting in

$$ii + \cos q\,(jj + kk) + \sin q\,(kj - jk)$$

for $\cos q$ and $\sin q$ their values in terms of $\tan\frac{1}{2}q$.

If we set, in either of these equations,

$$\theta = ai + bj + ck,$$

we obtain, on reduction, the formula

$$\Phi = \frac{\left\{\begin{array}{l}(1+a^2-b^2-c^2)ii + (2ab-2c)ij + (2ac+2b)ik \\ \ +(2ab+2c)ji + (1-a^2+b^2-c^2)jj + (2bc-2a)jk \\ \ +(2ac-2b)ki + (2bc+2a)kj + (1-a^2-b^2+c^2)kk\end{array}\right\}}{1+a^2+b^2+c^2},$$

in which the versor is expressed in terms of the rectangular components of the vector semitangent of version.

146. If α, β, γ are unit vectors, expressions of the form

$$2\alpha\alpha - I, \qquad 2\beta\beta - I, \qquad 2\gamma\gamma - I,$$

are biquadrantal versors. A product like

$$\{2\beta\beta - I\}.\{2\alpha\alpha - I\}$$

is a versor of which the axis is perpendicular to α and β, and the amount of rotation twice that which would carry α to β. It is evident that any versor may be thus expressed, and that either α or β may be given any direction perpendicular to the axis of rotation. If

$$\Phi = \{2\beta\beta - I\}.\{2\alpha\alpha - I\}, \quad \text{and} \quad \Psi = \{2\gamma\gamma - I\}.\{2\beta\beta - I\},$$

we have for the resultant of the successive rotations

$$\Psi.\Phi = \{2\gamma\gamma - I\}.\{2\alpha\alpha - I\}.$$

This may be applied to the composition of any two successive rotations, β being taken perpendicular to the two axes of rotation, and affords the means of determining the resultant rotation by construction on the surface of a sphere. It also furnishes a simple method of finding the relations of the vector semitangents of version for the versors Φ, Ψ, and $\Psi.\Phi$. Let

$$\theta_1 = \frac{-\Phi_\times}{1+\Phi_S}, \quad \theta_2 = \frac{-\Psi_\times}{1+\Psi_S}, \quad \theta_3 = \frac{-\{\Psi.\Phi\}_\times}{1+\{\Psi.\Phi\}_S}.$$

Then, since

$$\Phi = 4\alpha.\beta\beta\alpha - 2\alpha\alpha - 2\beta\beta + I,$$

$$\theta_1 = \frac{\alpha \times \beta}{\alpha.\beta},$$

which is moreover geometrically evident. In like manner,

$$\theta_2 = \frac{\beta \times \gamma}{\beta.\gamma}, \quad \theta_3 = \frac{\alpha \times \gamma}{\alpha.\gamma}.$$

Therefore

$$\begin{aligned}\theta_1 \times \theta_2 &= \frac{[\alpha \times \beta] \times [\beta \times \gamma]}{\alpha.\beta\,\beta.\gamma} = \frac{\alpha \times \beta.\gamma\beta}{\alpha.\beta\,\beta.\gamma} \\ &= \frac{\beta.\alpha\beta \times \gamma + \beta.\beta\gamma \times \alpha + \beta.\gamma\alpha \times \beta}{\alpha.\beta\,\beta.\gamma}\end{aligned}$$

(See No. 38.) That is,

$$\theta_1 \times \theta_2 = \theta_2 - \frac{\alpha.\gamma}{\alpha.\beta\,\beta.\gamma}\theta_3 + \theta_1.$$

Also,

$$\theta_1.\theta_2 = \frac{\alpha \times \beta.\beta \times \gamma}{\alpha.\beta\,\beta.\gamma} = 1 - \frac{\alpha.\gamma}{\alpha.\beta\,\beta.\gamma}.$$

Hence,

$$\theta_1 \times \theta_2 = \theta_2 - (1 - \theta_1.\theta_2)\theta_2 + \theta_1,$$

$$\theta_3 = \frac{\theta_1 + \theta_2 + \theta_2 \times \theta_1}{1 - \theta_1.\theta_2},$$

which is the formula for the composition of successive finite rotations by means of their vector semitangents of version.

147. The versors just described constitute a particular class under the more general form

$$\alpha\alpha' + \cos q\,\{\beta\beta' + \gamma\gamma'\} + \sin q\,\{\gamma\beta' - \beta\gamma'\},$$

in which α, β, γ are any non-complanar vectors, and α', β', γ' their reciprocals. A dyadic of this form as a prefactor does not affect any vector parallel to α. Its effect on a vector in the $\beta - \gamma$ plane will be best understood if we imagine an ellipse to be described of which β and γ are conjugate semi-diameters. If the vector to be operated on be a radius of this ellipse, we may evidently regard the ellipse with β, γ, and the other vector, as the projections of a circle with two perpendicular radii and one other radius. A little consideration will show that if the third radius of the circle is advanced an angle q, its projection in the ellipse will be advanced as required by the dyadic prefactor. The effect, therefore, of such a prefactor on a vector in the $\beta - \gamma$ plane may be obtained as follows: Describe an ellipse of which β and γ are conjugate semi-diameters. Then describe a similar and similarly placed ellipse of which the vector to be operated on is a radius. The effect of the operator is to advance the radius in this ellipse, in the angular direction from β toward γ, over a segment which is to the total area of the ellipse as q is to 2π. When used as a postfactor, the properties of the dyadic are similar, but the axis of no motion and the planes of rotation are in general different.

Def.—Such dyadics we shall call *cyclic*.

The Nth power (N being any whole number) of such a dyadic is obtained by multiplying q by N. If q is of the form $2\pi N/M$ (N and M being any whole numbers) the Mth power of the dyadic will be an idemfactor. A cyclic dyadic, therefore, may be regarded as a root of I, or at least capable of expression with any required degree of accuracy as a root of I.

It should be observed that the value of the above dyadic will not be altered by the substitution for α of any other parallel vector, or for β and γ of any other conjugate semi-diameters (which succeed one another in the same angular direction) of the same or any similar and similarly situated ellipse, with the changes which these substitutions require in the values of α', β', γ'. Or to consider the same changes from another point of view, the value of the dyadic will not be altered by the substitution for α' of any other parallel vector or for β' and γ' of any other conjugate semi-diameters (which succeed one another in the same angular direction) of the same or any similar and similarly situated ellipse, with the changes which these substitutions require in the values of α, β, and γ, defined as reciprocals of α', β', γ.

148. The strain represented by the equation

$$\rho' = \{aii + bjj + ckk\}.\rho$$

where a, b, c are positive scalars, may be described as consisting of three elongations (or contractions) parallel to the axes i, j, k, which are called the *principal axes of the strain*, and which have the property that their directions are not affected by the strain. The scalars a, b, c are called the *principal ratios of elongation*. (When one of these is less than unity, it represents a contraction.) The order of the three elongations is immaterial, since the original dyadic is equal to the product of the three dyadics

$$aii + jj + kk, \quad ii + bjj + kk, \quad ii + jj + ckk$$

taken in any order.

Def.—A dyadic which is reducible to this form we shall call a *right tensor*. The displacement represented by a right tensor is called a *pure strain*. A right tensor is evidently self-conjugate.

149. We have seen (No. 135) that every dyadic may be expressed in the form

$$\pm\{ai'i + bj'j + ck'k\},$$

where a, b, c are positive scalars. This is equivalent to

$$\pm\{ai'i' + bj'j' + ck'k'\}.\{i'i + j'j + k'k\}$$

and to

$$\pm\{i'i + j'j + k'k\}.\{aii + bjj + ckk\}.$$

Hence every dyadic may be expressed as the product of a versor and a right tensor with the scalar factor ± 1. The versor may precede or follow. It will be the same versor in either case, and the ratios of elongation will be the same; but the position of the principal axes of the tensor will differ in the two cases, either system being derived from the other by multiplication by the versor.

Def.—The displacement represented by the equation

$$\rho' = -\rho$$

is called *inversion*. The most general case of a homogeneous strain may therefore be produced by a pure strain and a rotation with or without inversion.

150. If

$$\Phi = ai'i + bj'j + ck'k,$$
$$\Phi.\Phi_C = a^2i'i' + b^2j'j' + c^2k'k',$$

and

$$\Phi_C.\Phi = a^2ii + b^2jj + c^2kk.$$

The general problem of the determination of the principal ratios and axes of strain for a given dyadic may thus be reduced to the case of a right tensor.

151. *Def.*—The effect of a prefactor of the form

$$a\alpha\alpha' + b\beta\beta' + c\gamma\gamma',$$

where a, b, c are positive or negative scalars, α, β, γ non-complanar vectors, and α', β', γ' their reciprocals, is to change α into $a\alpha$, β into $b\beta$, and γ into $c\gamma$. As a postfactor, the same dyadic will change α' into $a\alpha'$, β' into $b\beta'$, and γ' into $c\gamma'$. Dyadics which can be reduced to this form we shall call *tonic* (Gr. $\tau\epsilon\iota\nu\omega$). The right tensor already described constitutes a particular case, distinguished by perpendicular axes and positive values of the coefficients a, b, c.

The value of the dyadic is evidently not affected by substituting vectors of different lengths but the same or opposite directions for α, β, γ, with the necessary changes in the values of α', β', γ', defined as reciprocals of α, β, γ. But, except this change, if a, b, c are unequal, the dyadic can be expressed only in one way in the above form. If, however, two of these coefficients are equal, say a and b, any two non-collinear vectors in the $\alpha - \beta$ plane may be substituted for α and β, or, if the three coefficients are equal, any three non-complanar vectors may be substituted for α, β, γ.

152. Tonics having the same axes (determined by the directions of α, β, γ) are homologous, and their multiplication is effected by multiplying their coefficients. Thus,

$$\{a_1\alpha\alpha' + b_1\beta\beta' + c_1\gamma\gamma'\}.\{a_2\alpha\alpha' + b_2\beta\beta' + c_2\gamma\gamma'\}$$
$$= \{a_1a_2\alpha\alpha' + b_1b_2\beta\beta' + c_1c_2\gamma\gamma'\}.$$

Hence, division of such dyadics is effected by division of their coefficients. A tonic of which the three coefficients a, b, c are unequal, is homologous only with such dyadics as can be obtained by varying the coefficients.

153. The effect of a prefactor of the form

$$a\alpha\alpha' + b\{\beta\beta' + \gamma\gamma'\} + c\{\gamma\beta' - \beta\gamma'\},$$

or
$$a\alpha\alpha' + p\cos q\,\{\beta\beta' + \gamma\gamma'\} + p\sin q\,\{\gamma\beta' - \beta\gamma'\},$$

where α', β', γ' are the reciprocals of α, β, γ, and a, b, c, p, and q are scalars, of which p is positive, will be most evident if we resolve into the factors

$$a\alpha\alpha' + \beta\beta' + \gamma\gamma',$$
$$\alpha\alpha' + p\beta\beta' + p\gamma\gamma',$$
$$\alpha\alpha' + \cos q\,\{\beta\beta' + \gamma\gamma'\} + \sin q\,\{\gamma\beta' - \beta\gamma'\},$$

of which the order is immaterial, and if we suppose the vector on which we operate to be resolved into two factors, one parallel to α, and the other in the $\beta - \gamma$ plane. The effect of the first factor is to multiply by a the component parallel to α, without affecting the other. The effect of the second is to multiply by p the component in the $\beta - \gamma$ plane without affecting the other. The effect of the third is to give the component in the $\beta - \gamma$ plane the kind of elliptic rotation described in No. 147.

The effect of the same dyadic as a postfactor is of the same nature.

The value of the dyadic is not affected by the substitution for α of another vector having the same direction, nor by the substitution for β and γ of two other conjugate semi-diameters of the same or a similar and similarly situated ellipse, and which follow one another in the same angular direction.

Def.—Such dyadics we shall call *cyclotonic*.

154. Cyclotonics which are reducible to the same form except with respect to the values of a, p, and q are homologous. They are multiplied by multiplying the values of a, and also those of p, and adding those of q. Thus, the product of

$$a_1\alpha\alpha' + p_1 \cos q_1 \{\beta\beta' + \gamma\gamma'\} + p_1 \sin q_1 \{\gamma\beta' - \beta\gamma'\}$$

and
$$a_2\alpha\alpha' + p_2 \cos q_2 \{\beta\beta' + \gamma\gamma'\} + p_2 \sin q_2 \{\gamma\beta' - \beta\gamma'\}$$

is
$$\begin{aligned} a_1a_2\alpha\alpha' &+ p_1p_2 \cos(q_1+q_2) \{\beta\beta' + \gamma\gamma'\} \\ &+ p_1p_2 \sin(q_1+q_2) \{\gamma\beta' - \beta\gamma'\}. \end{aligned}$$

A dyadic of this form, in which the value of q is not zero, or the product of π and a positive or negative integer, is homologous only with such dyadics as are obtained by varying the values of a, p, and q.

155. In general, any dyadic may be reduced to the form either of a tonic or of a cyclotonic. (The exceptions are such as are made by the limiting cases.) We may show this, and also indicate how the reduction may be made, as follows. Let Φ be any dyadic. We have first to show that there is at least one direction of ρ for which

$$\Phi.\rho = a\rho.$$

This equation is equivalent to
$$\Phi.\rho - a\rho = 0,$$

or,
$$\{\Phi - aI\}.\rho = 0.$$

That is, $\Phi - aI$ is a planar dyadic, which may be expressed by the equation

$$|\Phi - aI| = 0.$$

(See No. 140.) Let
$$\Phi = \lambda i + \mu j + \nu k;$$

the equation becomes

$$|[\lambda - ai]i + [\mu - aj]j + [v - ak]k| = 0,$$

or,
$$[\lambda - ai] \times [\mu - aj].[\nu - ak] = 0,$$

or,
$$a^3 - (i.\lambda + j.\mu + k.\nu)a^2 + (i.\mu\times\nu + j.\nu\times\lambda + k.\lambda\times\mu)a - \lambda\times\mu.\nu = 0.$$

This may be written

$$a^3 - \Phi_S a^2 + \{\Phi^{-1}\}_S |\Phi| a - |\Phi| = 0.^2$$

Now if the dyadic Φ is given in any form, the scalars

$$\Phi_S, \quad \{\Phi^{-1}\}_S, \quad |\Phi|$$

are easily determined. We have therefore a cubic equation in a, for which we can find at least one and perhaps three roots. That is, we can find at least one value of a, and perhaps three, which will satisfy the equation

$$|\Phi - aI| = 0.$$

By substitution of such a value, $\Phi - aI$ becomes a planar dyadic, the planes of which may be easily determined.[3] Let α be a vector normal to the plane of the consequents. Then

$$\{\Phi - aI\}.\alpha = 0,$$

$$\Phi.\alpha = a\alpha.$$

If Φ is a tonic, we may obtain three equations of this kind, say

$$\Phi.\alpha = a\alpha, \quad \Phi.\beta = b\beta, \quad \Phi.\gamma = c\gamma,$$

in which α, β, γ are not complanar. Hence (by No. 108),

$$\Phi = a\alpha\alpha' + b\beta\beta' + c\gamma\gamma',$$

where α', β', γ' are the reciprocals of α, β, γ.

In any case, we may suppose a to have the same sign as $|\Phi|$, since the cubic equation must have such a root. Let α (as before) be normal to the plane of the consequents of the planar $\Phi - aI$, and α' normal to the plane of the antecedents, the lengths of α and α' being such that $\alpha.\alpha' = 1$.[4] Let β be any vector normal to α', and such that $\Phi.\beta$ is not parallel to β. (The case in which $\Phi.\beta$ is always parallel to β, if β is perpendicular to α', is evidently that of a tonic, and needs no farther discussion.) $\{\Phi - aI\}.\beta$ and therefore $\Phi.\beta$ will be perpendicular to α'. The same will be true of $\Phi^2.\beta$. Now (by No. 140)

$$[\Phi.a].[\Phi^2.\beta] \times [\Phi.\beta] = |\Phi|\alpha.[\Phi.\beta] \times \beta,$$

that is, $$a\alpha.[\Phi^2.\beta] \times [\Phi.\beta] = |\Phi|\alpha.[\Phi.\beta] \times \beta.$$

Hence, since $[\Phi^2.\beta] \times [\Phi.\beta]$ and $[\Phi.\beta] \times \beta$ are parallel,

$$a[\Phi^2.\beta] \times [\Phi.\beta] = |\Phi|[\Phi.\beta] \times \beta.$$

[2][See note at end of this paper.]

[3]In particular cases, $\Phi - aI$ may reduce to a linear dyadic, or to zero. These, however, will present no difficulties to the student.

[4]For the case in which the two planes are perpendicular to each other, see No. 157.

Since $a^{-1}|\Phi|$ is positive, we may set

$$p^2 = a^{-1}|\Phi|.$$

If we also set

$$\beta_1 = p^{-1}\Phi.\beta, \qquad \beta_2 = p^{-2}\Phi^2.\beta, \qquad \text{etc}.,$$
$$\beta_{-1} = p\Phi^{-1}.\beta, \qquad \beta_{-2} = p^2\Phi^{-2}.\beta, \qquad \text{etc}.,$$

the vectors β, β_1, β_2, etc., β_{-1}, β_{-2}, etc., will all lie in the plane perpendicular to α', and we shall have

$$\beta_2 \times \beta_1 = \beta_1 \times \beta,$$
$$[\beta_2 + \beta] \times \beta_1 = 0.$$

We may therefore set

$$\beta_2 + \beta = 2n\beta_1.$$

Multiplying by $p^{-1}\Phi$, and by $p\Phi^{-1}$,

$$\beta_3 + \beta_1 = 2n\beta_2, \qquad \beta_4 + \beta_2 = 2n\beta_3, \qquad \text{etc}.$$
$$\beta_1 + \beta_{-1} = 2n\beta, \qquad \beta + \beta_{-2} = 2n\beta_{-1}, \qquad \text{etc}.$$

Now, if $n > 1$, and we lay off from a common origin the vectors

$$\beta, \quad \beta_1, \quad \beta_2, \text{etc}., \qquad \beta_{-1}, \quad \beta_{-2}, \quad \text{etc}.,$$

the broken line joining the termini of these vectors will be convex toward the origin. All these vectors must therefore lie between two limiting lines, which may be drawn from the origin, and which may be described as having the directions of β_∞ and $\beta_{-\infty}$.[5] A vector having either of these directions is unaffected in direction by multiplication by Φ. In this case, therefore, Φ is a tonic. If $n < -1$ we may obtain the same result by considering the vectors

$$\beta, \quad -\beta_1, \quad \beta_2, \quad -\beta_2, \quad -\beta_3, \quad \beta_4, \quad \text{etc}., \qquad -\beta_{-1}, \quad \beta_{-2}, \quad -\beta_{-3}, \quad \text{etc}.$$

except that a vector in the limiting directions will be reversed in direction by multiplication by Φ, which implies that the two corresponding coefficients of the tonic are negative.

If $1 > n > -1$,[6] we may set

$$n = \cos q.$$

Then

$$\beta_{-1} + \beta_1 = 2\cos q\,\beta.$$

Let us now determine γ the equation

$$\beta_1 = \cos q\,\beta + \sin q\,\gamma.$$

This gives

$$\beta_{-1} = \cos q\,\beta - \sin q\,\gamma.$$

[5]The termini of the vectors will in fact lie on a hyperbola.

[6]For the limiting cases, in which $n = 1$, or $n = -1$, see No. 156.

Now α' is one of the reciprocals of α, β, and γ. Let β' and γ' be the others. If we set

$$\Psi = \cos q\,\{\beta\beta' + \gamma\gamma'\} + \sin q\,\{\gamma\beta' - \beta\gamma'\},$$

we have $$\Psi.\alpha = 0, \quad \Psi.\beta = \beta_1, \quad \Psi.\beta_{-1} = \beta.$$

Therefore, since

$$\begin{aligned}\{a\alpha\alpha' + p\Psi\}.\alpha &= a\alpha = \Psi.\alpha,\\ \{a\alpha\alpha' + p\Psi\}.\beta &= p\beta_1 = \Phi.\beta,\\ \{a\alpha\alpha' + p\Psi\}.\beta_{-1} &= p\beta = \Phi.\beta_{-1}.\end{aligned}$$

it follows (by No. 105) that

$$\Phi = a\alpha\alpha' + p\Psi = a\alpha\alpha' + p\cos q\,\{\beta\beta' + \gamma\gamma'\} + p\sin q\,\{\gamma\beta' - \beta\gamma'\}.$$

156. It will be sufficient to indicate (without demonstration) the forms of dyadics which belong to the particular cases which have been passed over in the preceding paragraph, so far as they present any notable peculiarities.

If $n = \pm 1$ (No. 155), the dyadic may be reduced to the form

$$a\alpha\alpha' + b\{\beta\beta' + \gamma\gamma'\} + bc\beta\gamma',$$

where α, β, γ are three non-complanar vectors, α', β', γ' their reciprocals, and a, b, c positive or negative scalars. The effect of this as an operator, will be evident if we resolve it into the three homologous factors

$$\begin{aligned}&a\alpha\alpha' + \beta\beta' + \gamma\gamma',\\ &\alpha\alpha' + b\{\beta\beta' + \gamma\gamma'\},\\ &\alpha\alpha' + \beta\beta' + \gamma\gamma' + c\beta\gamma'.\end{aligned}$$

The displacement due to the last factor may be called a *simple shear*. It consists (when the dyadic is used as prefactor) of a motion parallel to β, and proportioned to the distance from the $\alpha - \beta$ plane. This factor may be called a *shearer*.

This dyadic is homologous with such as are obtained by varying the values of a, b, c, and only with such, when the values of a and b are different, and that of c other than zero.

157. If the planar $\Phi - aI$ (No. 155) has perpendicular planes, there may be another value of a, of the same sign as $|\Phi|$, which will give a planar which has not perpendicular planes. When this is not the case, the dyadic may always be reduced to the form

$$a\{\alpha\alpha' + \beta\beta' + \gamma\gamma'\} + ab\{\alpha\beta' + \beta\gamma'\} + ac\alpha\gamma',$$

where α, β, γ are three non-complanar vectors, α', β', γ', their reciprocals, and a, b, c, positive or negative scalars. This may be resolved into the homologous factors

$$aI \quad \text{and} \quad I + b\{\alpha\beta' + \beta\gamma'\} + c\alpha\gamma'.$$

The displacement due to the last factor may be called a *complex shear*. It consists (when the dyadic is used as prefactor) of a motion parallel to α which is proportional to the distance from the $\alpha - \gamma$ plane, together with a motion parallel to $b\beta + c\alpha$ which is proportional to the distance from the $\alpha - \beta$ plane. This factor may be called a *complex shearer*.

This dyadic is homologous with such as are obtained by varying the values of a, b, c, and only such, unless $b = 0$.

It is always possible to take three mutually perpendicular vectors for α, β, and γ; or, if it be preferred, to take such values for these vectors as shall make the term containing c vanish.

158. The dyadics described in the two last paragraphs may be called *shearing dyadics*.

The criterion of a shearer is

$$\{\Phi - I\}^3 = 0, \qquad \Phi - I \neq 0.$$

The criterion of a simple shearer is

$$\{\Phi - I\}^2 = 0, \qquad \Phi - I \neq 0.$$

The criterion of a complex shearer is

$$\{\Phi - I\}^2 = 0, \qquad \{\Phi - I\}^2 \neq 0.$$

NOTE.—If a dyadic Φ is a linear function of a vector p (the term linear being used in the same sense as in No. 105), we may represent the relation by an equation of the form

$$\Phi = \alpha\beta\gamma.\rho + \epsilon\xi\eta.\rho + \text{etc}.,$$

or

$$\Phi = \{\alpha\beta\gamma + \epsilon\xi\eta + \text{etc}.\}.\rho,$$

where the expression in the braces may be called a *triadic polynomial*, and a single term $\alpha\beta\gamma$ a *triad*, or the indeterminate product of the three vectors α, β, γ. We are thus led successively to the consideration of higher orders of indeterminate products of vectors, *triads, tetrads*, in general *polyads*, and of polynomials consisting of such terms, *triadics, tetradics*, etc., in general *polyadics*. But the development of the subject in this direction lies beyond our present purpose.

It may sometimes be convenient to use notations like

$$\frac{\lambda, \mu, \nu}{|\alpha, \beta, \gamma} \quad \text{and} \quad \frac{\lambda, \mu, \nu}{\alpha, \beta, \gamma|}$$

to represent the conjugate dyadics which, the first as prefactor, and the second as postfactor, change α, β, γ into λ, μ, ν, respectively. In the notations of the preceding chapter these would be written

$$\lambda\alpha' + \mu\beta' + \nu\gamma' \quad \text{and} \quad \alpha'\lambda + \beta'\mu + \gamma'\nu$$

respectively, α', β', γ' denoting the reciprocals of α, β, γ. If τ is a linear function of ρ, the dyadics which as prefactor and postfactor change ρ into τ may be written respectively

$$\frac{\tau}{|\rho} \quad \text{and} \quad \frac{\tau}{\rho|}.$$

If τ is any function of ρ, the dyadics which as prefactor and postfactor change $d\rho$ into $d\tau$ may be written respectively

$$\frac{d\tau}{|d\rho} \quad \text{and} \quad \frac{d\tau}{d\rho|}.$$

In the notation of the following chapter the second of these (when ρ denotes a position-vector) would be written $\nabla\tau$. The triadic which as prefactor changes $d\rho$ into $\frac{d\tau}{|d\rho}$ may be written $\frac{d^2\tau}{|d\rho^2}$, and that which as postfactor changes $d\rho$ into $\frac{d\tau}{d\rho|}$ may be written $\frac{d^2\tau}{d\rho^2|}$. The latter would be written $\nabla\nabla\tau$ in the notations of the following chapter.

Chapter 4

CONCERNING THE DIFFERENTIAL AND INTEGRAL CALCULUS OF VECTORS.

(Supplementary to Chapter II.)

159. If ω is a vector having continuously varying values in space, and ρ the vector determining the position of a point, we may set

$$\rho = xi + yj + zk,$$
$$d\rho = dx\, i + dy\, j + dz\, k,$$

and regard ω as a function of ρ, or of x, y, and z. Then,

$$d\omega = dx\frac{d\omega}{dx} + dy\frac{d\omega}{dy} + dz\frac{d\omega}{dz},$$

that is, $$d\omega = d\rho.\left\{i\frac{d\omega}{dx} + j\frac{d\omega}{dy} + k\frac{d\omega}{dz}\right\}.$$

If we set $$\nabla\omega = i\frac{d\omega}{dx} + j\frac{d\omega}{dy} + k\frac{d\omega}{dz},$$

$$d\omega = d\rho.\nabla\omega.$$

Here ∇ stands for

$$i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}.$$

exactly as in No. 52, except that it is here applied to a vector and produces a dyadic, while in the former case it was applied to a scalar and produced a vector. The dyadic

$\nabla\omega$ represents the nine differential coefficients of the three components of ω with respect to x, y, and z, just as the vector ∇u (where u is a scalar function of ρ) represents the three differential coefficients of the scalar u with respect to x, y, and z.

It is evident that the expressions $\nabla.\omega$ and $\nabla \times \omega$ already defined (No. 54) are equivalent to $\{\nabla\omega\}_S$ and $\{\nabla\omega\}_\times$.

160. An important case is that in which the vector operated on is of the form ∇u. We have then

$$d\nabla u = d\rho.\nabla\nabla u,$$

where

$$\nabla\nabla u = \left\{\begin{array}{r} \dfrac{d^2u}{dx^2}ii + \dfrac{d^2u}{dx\,dy}ij + \dfrac{d^2u}{dx\,dz}ik \\ +\dfrac{d^2u}{dy\,dx}ji + \dfrac{d^2u}{dy^2}jj + \dfrac{d^2u}{dy\,dz}jk \\ +\dfrac{d^2u}{dz\,dx}ki + \dfrac{d^2u}{dz\,dy}kj + \dfrac{d^2u}{dz^2}kk \end{array}\right\}.$$

This dyadic, which is evidently self-conjugate, represents the six differential coefficients of the second order of u with respect to x, y, and z.[1]

161. The operators $\nabla\times$ and $\nabla.$ may be applied to dyadics in a manner entirely analogous to their use with scalars. Thus we may define $\nabla \times \Phi$ and $\nabla.\Phi$ by the equations

$$\nabla \times \Phi = i \times \frac{d\Phi}{dx} + j \times \frac{d\Phi}{dy} + k \times \frac{d\Phi}{dz}$$
$$\nabla.\Phi = i.\frac{d\Phi}{dx} + j.\frac{d\Phi}{dy} + k.\frac{d\Phi}{dz}.$$

Then, if

$$\Phi = \alpha i + \beta j + \gamma k,$$
$$\nabla \times \Phi = \nabla \times \alpha i + \nabla \times \beta j + \nabla \times \gamma k.$$
$$\nabla.\Phi = \nabla.\alpha i + \nabla.\beta j + \nabla.\gamma k.$$

Or, if
$$\Phi = i\alpha + j\beta + k\gamma,$$

$$\nabla \times \Phi = i\left[\frac{d\gamma}{dy} - \frac{d\beta}{dz}\right] + j\left[\frac{d\alpha}{dz} - \frac{d\gamma}{dx}\right] + k\left[\frac{d\beta}{dx} - \frac{d\alpha}{dy}\right],$$

$$\nabla.\Phi = \frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz}.$$

162. We may now regard $\nabla.\nabla$ in expressions like $\nabla.\nabla\omega$ as representing two successive operations, the result of which will be

$$\frac{d^2\omega}{dx^2} + \frac{d^2\omega}{dy^2} + \frac{d^2\omega}{dz^2}$$

[1] We might proceed to higher steps in differentiation by means of the triadics $\nabla\nabla\omega$, $\nabla\nabla\nabla u$, the tetradics $\nabla\nabla\nabla\omega$, $\nabla\nabla\nabla\nabla u$, etc. See note at end of No. 158. In like manner a dyadic function of position in space (Φ) might be differentiated by means of the triadic $\nabla\Phi$, the tetradic $\nabla\nabla\Phi$, etc.

in accordance with the definition of No. 70. We may also write $\nabla.\nabla\Phi$ for

$$\frac{d^2\Phi}{dx^2} + \frac{d^2\Phi}{dy^2} + \frac{d^2\Phi}{dz^2},$$

although in this case we cannot regard $\nabla.\nabla$ as representing two successive operations until we have defined $\nabla\Phi$.[2]

That $\nabla.\nabla\Phi = \nabla\nabla.\Phi - \nabla\times\nabla\times\Phi$ will be evident if we suppose Φ to be expressed in the form $\alpha i + \beta j + \gamma k$. (See No. 71.)

163. We have already seen that

$$u'' - u' = \int d\rho.\nabla u,$$

where u' and u'' denote the values of u at the beginning and the end of the line to which the integral relates. The same relation will hold for a vector; i.e.,

$$\omega'' - \omega' = \int d\rho.\nabla\omega.$$

164. The following equations between surface-integrals for a closed surface and volume-integrals for the space enclosed seem worthy of mention. One or two have already been given, and are here repeated for the sake of comparison.

$$\int\int d\sigma\, u = \int\int\int dv\, \nabla u, \tag{1}$$

$$\int\int d\sigma\, \omega = \int\int\int dv\, \nabla\omega, \tag{2}$$

$$\int\int d\sigma.\omega = \int\int\int dv\, \nabla.\omega, \tag{3}$$

$$\int\int d\sigma.\Phi = \int\int\int dv\, \nabla.\Phi, \tag{4}$$

$$\int\int d\sigma\times\omega = \int\int\int dv\, \nabla\times\omega, \tag{5}$$

$$\int\int d\sigma\times\Phi = \int\int\int dv\, \nabla\times\Phi. \tag{6}$$

It may aid the memory of the student to observe that the transformation may be effected in each case by substituting $\int\int\int dv\,\nabla$ for $\int\int d\sigma$.

165. The following equations between line-integrals for a closed line and surface-integrals for any surface bounded by the line, may also be mentioned. (One of these has already been given. See No. 60.)

$$\int d\rho\, u = \int\int d\sigma\times\nabla u, \tag{1}$$

$$\int d\rho\, \omega = \int\int d\sigma\times\nabla\omega, \tag{2}$$

[2] See footnote to No. 160.

$$\int d\rho.\omega = \iint d\sigma.\nabla \times \omega, \tag{3}$$

$$\int d\rho.\Phi = \iint d\sigma.\nabla \times \Phi, \tag{4}$$

$$\int d\rho \times \omega = \iint \nabla\omega.d\sigma - \iint d\sigma\, \nabla.\omega. \tag{5}$$

These transformations may be effected by substituting $\iint [d\sigma \times \nabla]$ for $\int d\rho$. The brackets are here introduced to indicate that the multiplication of $d\sigma$ with the i, j, k implied in ∇ is to be performed before any other multiplication which may be required by a subsequent sign. (This notation is not recommended for ordinary use, but only suggested as a mnemonic artifice.)

166. To the equations in No. 65 may be added many others, as

$$\begin{aligned}
\nabla[u\omega] &= \nabla u\omega + u\nabla\omega, && (1)\\
\nabla[\tau \times \omega] &= \nabla\tau \times \omega - \nabla\omega \times \tau, && (2)\\
\nabla \times [\tau \times \omega] &= \omega.\nabla\tau - \nabla.\tau\omega - \tau.\nabla\omega + \nabla.\omega\tau, && (3)\\
\nabla(\tau.\omega) &= \nabla\tau.\omega + \nabla\omega.\tau, && (4)\\
\nabla.\{\tau\omega\} &= \nabla.\tau\omega + \tau.\nabla\omega, && (5)\\
\nabla \times \{\tau\omega\} &= \nabla \times \tau\omega - \tau \times \nabla\omega, && (6)\\
\nabla.\{u\Phi\} &= \nabla u.\Phi + u\nabla.\Phi, && (7)
\end{aligned}$$

$$\text{etc}.$$

The principle in all these cases is that if we have one of the operators ∇, $\nabla.$, $\nabla\times$ prefixed to a product of any kind, and we make any transformation of the expression which would be allowable if the ∇ were a vector (viz., by changes in the order of the factors, in the signs of multiplication, in the parentheses written or implied, etc.), by which changes the ∇ is brought into connection with one particular factor, the expression thus transformed will represent the part of the value of the original expression which results from the variation of that factor.

167. From the relations indicated in the last four paragraphs, may be obtained directly a great number of transformations of definite integrals similar to those given in Nos. 74-77, and corresponding to those known in the scalar calculus by the name of *integration by parts*.

168. The student will now find no difficulty in generalizing the integrations of differential equations given in Nos. 78-89 by applying to vectors those which relate to scalars, and to dyadics those which relate to vectors.

169. The propositions in No. 90 relating to minimum values of the volume-integral $\iiint u\omega.\omega\, dv$ may be generalized by substituting $\omega.\Phi.\omega$ for $u\omega.\omega$, Φ being a given dyadic function of position in space.

170. The theory of the integrals which have been called potentials, Newtonians, etc. (see Nos. 91-102) may be extended to cases in which the operand is a vector instead of a scalar or a dyadic instead of a vector. So far as the demonstrations are concerned, the case of a vector may be reduced to that of a scalar by considering separately its three

components, and the case of a dyadic may be reduced to that of a vector, by supposing the dyadic expressed in the form $\phi i + \chi j + \omega k$ and considering each of these terms separately.

Chapter 5

CONCERNING TRANSCENDENTAL FUNCTIONS OF DYADICS.

171. *Def.*—The *exponential function*, the *sine* and the *cosine* of a dyadic may be defined by infinite series, exactly as the corresponding functions in scalar analysis, viz.,

$$e^{\Phi} = I + \Phi + \frac{1}{2}\Phi^2 + \frac{1}{2.3}\Phi^3 + \text{etc}.,$$
$$\sin\Phi = \Phi - \frac{1}{2.3}\Phi^3 + \frac{1}{2.3.4.5}\Phi^5 - \text{etc}.,$$
$$\cos\Phi = I - \frac{1}{2}\Phi^2 + \frac{1}{2.3.4}\Phi^4 - \text{etc}.$$

These series are always convergent. For every value of Φ there is one and only one value of each of these functions. The exponential function may also be defined as the limit of the expression

$$\left(I + \frac{\Phi}{N}\right)^N,$$

when N, which is a whole number, is increased indefinitely. That this definition is equivalent to the preceding, will appear if the expression is expanded by the binomial theorem, which is evidently applicable in a case of this kind.

These functions of Φ are homologous with Φ.

172. We may define the logarithm as the function which is the inverse of the exponential, so that the equations

$$e^{\Psi} = \Phi,$$
$$\Psi = \log\Phi,$$

are equivalent, leaving it undetermined for the present whether every dyadic has a logarithm, and whether a dyadic can have more than one.

173. It follows at once from the second definition of the exponential function that, if Φ and Ψ are homologous,

$$e^{\Phi}.e^{\Psi} = e^{\Phi+\Psi},$$

and that, if T is a positive or negative whole number,

$$\{e^{\Phi}\}^T = e^{T\Phi}.$$

174. If Ξ and Φ are homologous dyadics, and such that

$$\Xi^2.\Phi = -\Phi,$$

the definitions of No. 171 give immediately

$$\begin{aligned} e^{\Xi.\Phi} &= \cos\Phi + \Xi\sin\Phi, \\ e^{-\Xi.\Phi} &= \cos\Phi - \Xi\sin\Phi, \end{aligned}$$

whence

$$\begin{aligned} \cos\Phi &= \frac{1}{2}\{e^{\Xi\Phi} + e^{-\Xi\Phi}\}, \\ \sin\Phi &= -\frac{1}{2}\Xi\{e^{\Xi\Phi} - e^{-\Xi\Phi}\}. \end{aligned}$$

175. If $\Phi.\Psi = \Psi.\Phi = 0$,

$$\{\Phi+\Psi\}^2 = \Phi^2 + \Psi^2, \qquad \{\Phi+\Psi\}^3 = \Phi^3 + \Psi^3, \quad \text{etc}\,.$$

Therefore

$$\begin{aligned} e^{\Phi+\Psi} &= e^{\Phi} + e^{\Psi} - I, \\ \cos\{\Phi+\Psi\} &= \cos\Phi + \cos\Psi - I, \\ \sin\{\Phi+\Psi\} &= \sin\Phi + \sin\Psi. \end{aligned}$$

176.

$$|e^{\Phi}| = e^{\Phi_S}.$$

For the first member of this equation is the limit of

$$|\{I + N^{-1}\Phi\}^N|,$$

that is, of

$$|I + N^{-1}\Phi|^N.$$

If we set $\Phi = \alpha i + \beta j + \gamma k$, the limit becomes that of

$$(1 + N^{-1}\alpha.i + N^{-1}\beta.j + N^{-1}\gamma.k)^N, \quad \text{or} \quad (1 + N^{-1}\Phi_S)^N,$$

the limit of which is the second member of the equation to be proved.

177. By the definition of exponentials, the expression

$$e^{q\{kj-jk\}}$$

represents the limit of

$$\{I + qN^{-1}\{kj - jk\}\}^N.$$

Now $I + qN^{-1}\{kj - jk\}$ evidently represents a versor having the axis i and the infinitesimal angle of version qN^{-1}. Hence the above exponential represents a versor having the same axis and the angle of version q. If we set $qi = \omega$, the exponential may be written

$$e^{I \times \omega}.$$

Such an expression therefore represents a versor. The axis and direction of rotation are determined by the direction of ω, and the angle of rotation is equal to the magnitude of ω. The value of the versor will not be affected by increasing or diminishing the magnitude of ω by 2π.

178. If, as in No. 151,

$$\Phi = a\alpha\alpha' + b\beta\beta' + c\gamma\gamma',$$

the definitions of No. 171 give

$$\begin{aligned} e^{\Phi} &= e^a\alpha\alpha' + e^b\beta\beta' + e^c\gamma\gamma', \\ \cos\Phi &= \cos a\alpha\alpha' + \cos b\beta\beta' + \cos c\gamma\gamma', \\ \sin\Phi &= \sin a\alpha\alpha' + \sin b\beta\beta' + \sin c\gamma\gamma'. \end{aligned}$$

If a, b, c are positive and unequal, we may add, by No. 172,

$$\log\Phi = \log a\alpha\alpha' + \log b\beta\beta' + \log c\gamma\gamma'.$$

179. If, as in No. 153,

$$\begin{aligned} \Phi &= a\alpha\alpha' + b\{\beta\beta' + \gamma\gamma'\} + c\{\gamma\beta' - \beta\gamma'\} \\ &= a\alpha\alpha' + p\cos q\,\{\beta\beta' + \gamma\gamma'\} + p\sin q\,\{\gamma\beta' - \beta\gamma'\}, \end{aligned}$$

we have by No. 173

$$e^{\Phi} = e^{a\alpha\alpha'}.e^{b\{\beta\beta'+\gamma\gamma'\}}.e^{c\{\gamma\beta'-\beta\gamma'\}}.$$

But

$$\begin{aligned} e^{a\alpha\alpha'} &= e^a\alpha\alpha' + \beta\beta' + \gamma\gamma', \\ e^{b\{\beta\beta'+\gamma\gamma'\}} &= \alpha\alpha' + e^b\{\beta\beta' + \gamma\gamma'\}, \\ e^{c\{\gamma\beta'-\beta\gamma'\}} &= \alpha\alpha' + \cos c\,\{\beta\beta' + \gamma\gamma'\} + \sin c\,\{\gamma\beta' - \beta\gamma'\}. \end{aligned}$$

Therefore,

$$e^{\Phi} = e^a\alpha\alpha' + e^b\cos c\,\{\beta\beta' + \gamma\gamma'\} + e^b\sin c\,\{\gamma\beta' - \beta\gamma'\}.$$

Hence, if a is positive,

$$\log\Phi = \log a\alpha\alpha' + \log p\{\beta\beta' + \gamma\gamma'\} + q\{\gamma\beta' - \beta\gamma'\}.$$

Since the value of Φ is not affected by increasing or diminishing q by 2π, the function $\log\Phi$ is many-valued.

To find the value of $\cos\Phi$ and $\sin\Phi$ let us set

$$\Theta = b\{\beta\beta' + \gamma\gamma'\} + c\{\gamma\beta' - \beta\gamma'\},$$

$$\Xi = \gamma\beta' - \beta\gamma'.$$

Then, by No, 175,

$$\cos\Phi = \cos\{a\alpha\alpha'\} + \cos\Theta - I.$$

But

$$\cos\{a\alpha\alpha'\} - I = \cos a\alpha\alpha' - \alpha\alpha'.$$

Therefore,

$$\cos\Phi = \cos a\alpha\alpha' - \alpha\alpha' + \cos\Theta.$$

Now, by No. 174,

$$\cos\Theta = \frac{1}{2}\{e^{\Xi.\Theta} + e^{-\Xi.\Theta}\}.$$

Since

$$\begin{aligned}
\Xi.\Theta &= -c\{\beta\beta' + \gamma\gamma'\} + b\{\gamma\beta' - \beta\gamma'\},\\
e^{\Xi.\Theta} &= \alpha\alpha' + e^{-c}\cos b\,\{\beta\beta' + \gamma\gamma'\} + e^{-c}\sin b\,\{\gamma\beta' - \beta\gamma'\},\\
e^{-\Xi.\Theta} &= \alpha\alpha' + e^{c}\cos b\,\{\beta\beta' + \gamma\gamma'\} - e^{c}\sin b\,\{\gamma\beta' - \beta\gamma'\}.
\end{aligned}$$

Therefore

$$\cos\Theta = \alpha\alpha' + \frac{1}{2}(e^c + e^{-c})\cos b\,\{\beta\beta' + \gamma\gamma'\} - \frac{1}{2}(e^c - e^{-c})\sin b\,\{\gamma\beta' - \beta\gamma'\},$$

and

$$\cos\Phi = \cos a\alpha\alpha' + \frac{1}{2}(e^c + e^{-c})\cos b\,\{\beta\beta' + \gamma\gamma'\} - \frac{1}{2}(e^c - e^{-c})\sin b\,\{\gamma\beta' - \beta\gamma'\}.$$

In like manner we find

$$\sin\Phi = \sin a\alpha\alpha' + \frac{1}{2}(e^c + e^{-c})\sin b\,\{\beta\beta' + \gamma\gamma'\} + \frac{1}{2}(e^c - e^{-c})\cos b\,\{\gamma\beta' - \beta\gamma'\}.$$

180. If α, β, γ and α', β', γ' are reciprocals, and

$$\Phi = a\alpha\alpha' + b\{\beta\beta' + \gamma\gamma'\} + c\beta\gamma',$$

and N is any whole number,

$$\Phi^N = a^N\alpha\alpha' + b^N\{\beta\beta' + \gamma\gamma'\} + e^b c\beta\gamma'.$$

Therefore,

$$\begin{aligned}
e^\Phi &= e^a\alpha\alpha' + e^b\{\beta\beta' + \gamma\gamma'\} + e^b c\beta\gamma',\\
\cos\Phi &= \cos a\alpha\alpha' + \cos b\,\{\beta\beta' + \gamma\gamma'\} - c\sin b\,\beta\gamma'\\
\sin\Phi &= \sin a\alpha\alpha' + \sin b\,\{\beta\beta' + \gamma\gamma'\} + c\cos b\,\beta\gamma'
\end{aligned}$$

If a and b are unequal, and c other than zero, we may add

$$\log \Phi = \log a\alpha\alpha' + \log b\{\beta\beta' + \gamma\gamma'\} + cb^{-1}\beta\gamma.$$

181. If α, β, γ, and α', β', γ' are reciprocals, and

$$\Phi = aI + b\{\alpha\beta' + \beta\gamma'\} + c\alpha\gamma',$$

and N is a whole number,

$$\Phi^N = a^N I + Na^{N-1}b\{\alpha\beta' + \beta\gamma'\} + (Na^{N-1}c + \frac{1}{2}N(N-1)a^{N-2}b^2)\alpha\gamma'.$$

Therefore

$$e^\Phi = e^a I + e^a b\{\alpha\beta' + \beta\gamma'\} + e^a(\frac{1}{2}b^2 + c)\alpha\gamma',$$

$$\cos\Phi = \cos aI - b\sin a\,\{\alpha\beta' + \beta\alpha'\} - (\frac{1}{2}b^2\cos a + c\sin a)\alpha\gamma',$$

$$\sin\Phi = \sin aI + b\cos a\,\{\alpha\beta' + \beta\alpha'\} - (\frac{1}{2}b^2\sin a - c\cos a)\alpha\gamma'.$$

Unless $b = 0$, we may add

$$\log\Phi = \log aI + ba^{-1}\{\alpha\beta' + \beta\alpha'\} + (ca^{-1} - \frac{1}{2}b^2a^{-2})\alpha\gamma'.$$

182. If we suppose any dyadic Φ to vary, but with the limitation that all its values are homologous, we may obtain from the definitions of No. 171

$$\begin{aligned} d\{e^\Phi\} &= e^\Phi.d\Phi, && (1)\\ d\sin\Phi &= \cos\Phi.d\Phi, && (2)\\ d\cos\Phi &= -\sin\Phi.d\Phi, && (3)\\ d\log\Phi &= \Phi^{-1}.d\Phi, && (4) \end{aligned}$$

as in the ordinary calculus, but we must not apply these equations to cases in which the values of Φ are not homologous.

183. If, however, Γ is any constant dyadic, the variations of $t\Gamma$ will necessarily be homologous with $t\Gamma$, and we may write without other limitation than that Γ is constant,

$$\begin{aligned} \frac{d\{e^{t\Gamma}\}}{dt} &= \Gamma.e^{t\Gamma} && (1)\\ \frac{d\sin\{t\Gamma\}}{dt} &= \Gamma.\cos\{t\Gamma\}, && (2)\\ \frac{d\cos\{t\Gamma\}}{dt} &= -\Gamma.\sin\{t\Gamma\}, && (3)\\ \frac{d\log\{t\Gamma\}}{dt} &= \frac{I}{t}. && (4) \end{aligned}$$

A second differentiation gives

$$\frac{d^2\{e^{t\Gamma}\}}{dt^2} = \Gamma^2.e^{t\Gamma}, \tag{5}$$

$$\frac{d^2\sin\{t\Gamma\}}{dt^2} = -\Gamma^2.\sin\{t\Gamma\}, \tag{6}$$

$$\frac{d^2\cos\{t\Gamma\}}{dt^2} = -\Gamma^2.\cos\{t\Gamma\}. \tag{7}$$

184. It follows that if we have a differential equation of the form

$$\frac{d\rho}{dt} = \Gamma.\rho,$$

the integral equation will be of the form

$$\rho = e^{t\Gamma}.\rho',$$

ρ' representing the value of ρ for $t = 0$. For this gives

$$\frac{d\rho}{dt} = \Gamma.e^{t\Gamma}.\rho' = \Gamma.\rho,$$

and the proper value of ρ for $t = 0$.

185. *Def.*—A flux which is a linear function of the position-vector is called a *homogeneous-strain-flux* from the nature of the strain which it produces. Such a flux may evidently be represented by a dyadic.

In the equations of the last paragraph, we may suppose ρ to represent a position-vector, t the time, and Γ a homogeneous-strain-flux. Then $e^{t\Gamma}$ will represent the strain produced by the flux Γ in the time t.

In like manner, if Λ represents a homogeneous strain, $\{\log\Lambda\}/t$ will represent a homogeneous-strain-flux which would produce the strain Λ in the time t.

186. If we have

$$\frac{d^2\rho}{dt^2} = \Gamma^2.\rho,$$

where Γ is complete, the integral equation will be of the form

$$\rho = e^{t\Gamma}.\alpha + e^{-t\Gamma}.\beta.$$

For this gives

$$\frac{d\rho}{dt} = \Gamma.e^{t\Gamma}.\alpha - \Gamma.e^{-t\Gamma}.\beta,$$

$$\frac{d^2\rho}{dt^2} = \Gamma^2.e^{t\Gamma}.\alpha + \Gamma^2.e^{-t\Gamma}.\beta = \Gamma^2.\rho,$$

and α and β may be determined so as to satisfy the equations

$$\rho_{t=0} = \alpha + \beta,$$

$$\left[\frac{d\rho}{dt}\right]_{t=0} = \Gamma.\{\alpha - \beta\}.$$

187. The differential equation

$$\frac{d^2\rho}{dt^2} = -\Gamma^2.\rho$$

will be satisfied by

$$\rho = \cos\{t\Gamma\}.\alpha + \sin\{t\Gamma\}.\beta,$$

whence

$$\frac{d\rho}{dt} = -\Gamma.\sin\{t\Gamma\}.\alpha + \Gamma.\cos\{t\Gamma\}.\beta,$$

$$\frac{d^2\rho}{dt^2} = -\Gamma^2.\cos\{t\Gamma\}.\alpha - \Gamma^2.\sin\{t\Gamma\}.\beta = -\Gamma^2.\rho.$$

If Γ is complete, the constants α and β may be determined to satisfy the equations

$$\rho_{t=0} = \alpha,$$

$$\left[\frac{d\rho}{dt}\right]_{t=0} = \Gamma.\beta.$$

188. If

$$\frac{d^2\rho}{dt^2} = \{\Gamma^2 - \Lambda^2\}.\rho,$$

where $\Gamma^2 - \Lambda^2$ is a complete dyadic, and

$$\Gamma.\Lambda = \Lambda.\Gamma = 0,$$

we may set

$$\rho = \{\frac{1}{2}e^{t\Gamma} + \frac{1}{2}e^{-t\Gamma} + \cos\{t\Lambda\} - I\}.\alpha + \{\frac{1}{2}e^{t\Gamma} - \frac{1}{2}e^{-t\Gamma} + \sin\{t\Lambda\}\}.\beta,$$

which gives

$$\frac{d\rho}{dt} = \{\frac{1}{2}\Gamma.e^{t\Gamma} - \frac{1}{2}\Gamma.e^{-t\Gamma} - \Lambda.\sin\{t\Lambda\}\}.\alpha + \{\frac{1}{2}\Gamma.e^{t\Gamma} + \frac{1}{2}\Gamma.e^{-t\Gamma} + \Lambda.\cos\{t\Lambda\}\}.\beta,$$

$$\frac{d^2\rho}{dt^2} = \{\frac{1}{2}\Gamma^2.e^{t\Gamma} + \frac{1}{2}\Gamma^2.e^{-t\Gamma} - \Lambda^2.\cos\{t\Lambda\}\}.\alpha + \{\frac{1}{2}\Gamma^2.e^{t\Gamma} - \frac{1}{2}\Gamma^2.e^{-t\Gamma} - \Lambda^2.\sin\{t\Lambda\}\}.\beta$$
$$= \{\Gamma^2 - \Lambda^2\}.\rho.$$

The constants α and β are to be determined by

$$\rho_{t=0} = \alpha,$$

$$\left[\frac{d\rho}{dt}\right]_{t=0} = \{\Gamma + \Lambda\}.\beta.$$

189. It will appear, on reference to Nos. 155-157, that every complete dyadic may be expressed in one of three forms, viz., as a square, as a square with the negative sign, or as a difference of squares of two dyadics of which both the direct products are equal to zero. It follows that every equation of the form

$$\frac{d^2\rho}{dt^2} = \Theta.\rho,$$

where Θ is any constant and complete dyadic, may be integrated by the preceding formulæ.

Chapter 6

NOTE ON BIVECTOR ANALYSIS.

[1]

1. A vector is determined by three algebraic quantities. It often occurs that the solution of the equations by which these are to be determined gives imaginary values, i.e., instead of scalars we obtain biscalars, or expressions of the form $a + \iota b$, where a and b are scalars, and $\iota = \sqrt{-1}$. It is most simple, and always allowable, to consider the vector as determined by its components parallel to a normal system of axes. In other words, a vector may be represented in the form

$$xi + yj + zk.$$

Now if the vector is required to satisfy certain conditions, the solution of the equations which determine the values of x, y, and z, in the most general case, will give results of the form

$$x = x_1 + \iota x_2,$$
$$y = y_1 + \iota y_2,$$
$$z = z_1 + \iota z_2,$$

where x_1, x_2, y_1, y_2, z_1, z_2 are scalars. Substituting these values in

$$xi + yj + zk,$$

[1]Thus far, in accordance with the purpose expressed in the footnote to No. 1, we have considered only real values of scalars and vectors. The object of this limitation has been to present the subject in the most elementary manner. The limitation is however often inconvenient, and does not allow the most symmetrical and complete development of the subject in many important directions. Thus in Chapter V, and the latter part of Chapter III, the exclusion of imaginary values has involved a considerable sacrifice of simplicity both in the enunciation of theorems and in their demonstration. The student will find an interesting and profitable exercise in working over this part of the subject with the aid of imaginary values, especially in the discussion of the imaginary roots of the cubic equation of No. 155, and in the use of the formula

$$e^{\iota\Phi} = \cos\Phi + \iota\sin\Phi$$

in developing the properties of the *sines, cosines*, and *exponentials* of dyadics.

we obtain

$$(x_1 + \iota x_2)i + (y_1 + \iota y_2)j + (z_1 + \iota z_2)k;$$

or, if we set

$$\rho_1 = x_1 i + y_1 j + z_1 k,$$
$$\rho_2 = x_2 i + y_2 j + z_2 k,$$

we obtain

$$\rho_1 + \iota\rho_2.$$

We shall call this a *bivector*, a term which will include a vector as a particular case. When we wish to express a bivector by a single letter, we shall use the small German letters. Thus we may write

$$\mathfrak{r} = \rho_1 + \iota\rho_2.$$

An important case is that in which ρ_1 and ρ_2 have the same direction. The bivector may then be expressed in the form $(a + \iota b)\rho$, in which the vector factor, if we choose, may be a unit vector. In this case, we may say that the bivector has a *real direction*. In fact, if we express the bivector in the form

$$(x_1 + \iota x_2)i + (y_1 + \iota y_2)j + (z_1 + \iota z_2)k$$

the ratios of the coefficients of i, j, and k, which determine the direction cosines of the vector, will in this case be real.

2. The consideration that operations upon bivectors may be regarded as operations upon their biscalar x-, y-, and z-components is sufficient to show the possibility of a bivector analysis and to indicate what its rules must be. But this point of view does not afford the most simple conception of the operations which we have to perform upon bivectors. It is desirable that the definitions of the fundamental operations should be independent of such extraneous considerations as any system of axes.

The various signs of our analysis, when applied to bivectors, may therefore be defined as follows, viz.,

The bivector equation

$$\mu' + \iota\nu' = \mu'' + \iota\nu''$$

implies the two vector equations

$$\mu' = \mu'', \quad \text{and} \quad \nu' = \nu''.$$
$$-[\mu + \iota\nu] = -\mu + \iota[-\nu].$$
$$[\mu' + \iota\nu'] + [\mu'' + \iota\nu''] = [\mu' + \mu''] + \iota[\nu' + \nu''].$$
$$[\mu' + \iota\nu'].[\mu'' + \iota\nu''] = [\mu'.\mu'' - \nu'.\nu''] + \iota[\mu'.\nu'' + \nu'.\mu''].$$
$$[\mu' + \iota\nu'] \times [\mu'' + \iota\nu''] = [\mu' \times \mu'' - \nu' \times \nu''] + \iota[\mu' \times \nu'' + \nu' \times \mu''].$$[2]

---

[2]

$$(a + \iota b)[\mu + \iota\nu] = a\mu - b\nu + \iota[a\nu + b\mu].$$
$$[\mu + \iota\nu](a + \iota b) = \mu a - \nu b + \iota[\mu b + \nu a].$$

Therefore the position of the scalar factor is indifferent. [MS. note by author.]

With these definitions, a great part of the laws of vector analysis may be applied at once to bivector expressions. But an equation which is impossible in vector analysis may be possible in bivector analysis, and in general the number of roots of an equation, or of the values of a function, will be different according as we recognize, or do not recognize, imaginary values.

3. *Def.*—Two bivectors, or two biscalars, are said to be *conjugate*, when their real parts are the same, and their imaginary parts differ in sign, and in sign only.

Hence, the product of the conjugates of any number of bivectors and biscalars is the conjugate of the product of the bivectors and biscalars. This is true of any kind of product.

The products of a vector and its conjugate are as follows:

$$[\mu+\iota\nu].[\mu-\iota\nu]=\mu.\mu+\nu.\nu$$
$$[\mu+\iota\nu]\times[\mu-\iota\nu]=2\iota\nu\times\mu$$
$$[\mu+\iota\nu][\mu-\iota\nu]=\{\mu\mu+\nu\nu\}+\iota\{\nu\mu-\mu\nu\}.$$

Hence, if μ and $\iota\nu$ represent the real and imaginary parts of a bivector, the values of

$$\mu.\mu+\nu.\nu, \qquad \mu\times\nu, \qquad \mu\mu+\nu\nu, \qquad \nu\mu-\mu\nu,$$

are not affected by multiplying the bivector by a biscalar of the form $a+\iota b$, in which $a^2+b^2=1$, say a *cyclic scalar*. Thus, if we set

$$\mu'+\iota\nu'=(a+\iota b)[\mu+\iota\nu],$$

we shall have

$$\mu'-\iota\nu'=(a-\iota b)[\mu-\iota\nu],$$

and

$$[\mu'+\iota\nu'].[\mu'-\iota\nu']=[\mu+\iota\nu].[\mu-\iota\nu]$$

That is,

$$\mu'.\mu'+\nu'.\nu'=\mu.\mu+\nu\nu;$$

and so in the other cases.

4. *Def.*—In biscalar analysis, the product of a biscalar and its conjugate is a positive scalar. The positive square root of this scalar is called the modulus of the biscalar. In bivector analysis, the direct product of a bivector and its conjugate is, as seen above, a positive scalar. The positive square root of this scalar may be called the *modulus* of the *bivector*. When this modulus vanishes, the bivector vanishes, and only in this case. If the bivector is multiplied by a biscalar, its modulus is multiplied by the modulus of the biscalar. The conjugate of a (real) vector is the vector itself, and the modulus of the vector is the same as its magnitude.

5. *Def.*—If between two vectors, α and β, there subsists a relation of the form

$$\alpha=n\beta,$$

where n is a scalar, we say that the vectors are parallel. Analogy leads us to call two bivectors *parallel*, when there subsists between them a relation of the form

$$\mathfrak{a}=m\,\mathfrak{b}$$

where m (in the most general case) is a biscalar.

To aid us in comprehending the geometrical signification of this relation, we may regard the biscalar as consisting of two factors, one of which is a positive scalar (the modulus of the biscalar), and the other may be put in the form $\cos q + \iota \sin q$. The effect of multiplying a bivector by a positive scalar is obvious. To understand the effect of a multiplier of the form $\cos q + \iota \sin q$ upon a bivector $\mu + \iota\nu$, let us set

$$\mu' + \iota\nu' = (\cos q + \iota \sin q)[\mu + \iota\nu].$$

We have then

$$\begin{aligned} \mu' &= \cos q\, \mu - \sin q\, \nu, \\ \nu' &= \cos q\, \nu + \sin q\, \mu. \end{aligned}$$

Now if μ and ν are of the same magnitude and at right angles, the effect of the multiplication is evidently to rotate these vectors in their plane an angular distance q, which is to be measured in the direction from ν to μ. In any case we may regard μ and ν as the projections (by parallel lines) of two perpendicular vectors of the same length. The two last equations show that μ' and ν' will be the projections of the vectors obtained by the rotation of these perpendicular vectors in their plane through the angle q. Hence, if we construct an ellipse of which μ and ν are conjugate semi-diameters, μ' and ν' will be another pair of conjugate semi-diameters, and the sectors between μ and μ' and between ν and ν', will each be to the whole area of the ellipse as q to 2π, the sector between ν and ν' lying on the same side of ν and μ, and that between μ and μ' lying on the same side of μ as $-\nu$.

It follows that any bivector $\mu + \iota\nu$ may be put in the form

$$(\cos q + \iota \sin q)[\alpha + \iota\beta],$$

in which α and β are at right angles, being the semi-axes of the ellipse of which μ and ν are conjugate semi-diameters. This ellipse we may call the *directional ellipse* of the bivector. In the case of a real vector, or of a vector having a real direction, it reduces to a straight line. In any other case, the angular direction from the imaginary to the real part of the bivector is to be regarded as positive in the ellipse, and the specification of the ellipse must be considered incomplete without the indication of this direction.

Parallelism of bivectors, then, signifies the similarity and similar position of their directional ellipses. Similar position includes identity of the angular directions mentioned above.

6. To reduce a given bivector $\mathfrak{r}$ to the above form, we may set

$$\begin{aligned} \mathfrak{r} \,.\, \mathfrak{r} &= (\cos q + \iota \sin q)^2 \, [\alpha + \iota\beta].[\alpha + \iota\beta] \\ &= (\cos 2q + \iota \sin 2q)(\alpha.\alpha - \beta.\beta) \\ &= a + \iota b, \end{aligned}$$

where a and b are scalars, which we may regard as known. The value of q may be determined by the equation

$$\tan 2q = \frac{b}{a},$$

the quadrant to which $2q$ belongs being determined so as to give $\sin 2q$ and $\cos 2q$ the same signs as b and a. Then α and β will be given by the equation

$$\alpha + \iota\beta = (\cos q - \iota \sin q)\, \mathfrak{r}\,.$$

The solution is indeterminate when the real and imaginary parts of the given bivector are perpendicular and equal in magnitude. In this case the directional ellipse is a circle, and the bivector may be called *circular*. The criterion of a circular bivector is

$$\mathfrak{r} . \mathfrak{r} = 0.$$

It is especially to be noticed that from this equation we cannot conclude that

$$\mathfrak{r} = 0,$$

as in the analysis of real vectors. This may also be shown by expressing $\mathfrak{r}$ in the form $xi + yj + zk$, in which x, y, z are biscalars. The equation then becomes

$$x^2 + y^2 + z^2 = 0,$$

which evidently does not require x, y, and z to vanish, as would be the case if only real values are considered

7. *Def.*—We call two vectors ρ and σ perpendicular when $\rho.\sigma = 0$. Following the same analogy, we shall call two bivectors $\mathfrak{r}$ and $\mathfrak{s}$ perpendicular, when

$$\mathfrak{r} . \mathfrak{s} = 0.$$

In considering the geometrical signification of this equation, we shall first suppose that the real and imaginary components of $\mathfrak{r}$ and $\mathfrak{s}$ lie in the same plane, and that both $\mathfrak{r}$ and $\mathfrak{s}$ have not real directions. It is then evidently possible to express them in the form

$$m[\alpha + \iota\beta], \qquad m'[\alpha' + \iota\beta'],$$

where m and m' are biscalar, α and β are at right angles, and α' parallel with β. Then the equation $\mathfrak{r} . \mathfrak{s} = 0$ requires that

$$\beta.\beta' = 0, \quad \text{and} \quad \alpha.\beta' + \beta.\alpha' = 0.$$

This shows that the directional ellipses of the two bivectors are similar and the angular direction from the real to the imaginary component is the same in both, but the major axes of the ellipses are perpendicular. The case in which the directions of $\mathfrak{r}$ and $\mathfrak{s}$ are real, forms no exception to this rule.

It will be observed that every circular bivector is perpendicular to itself, and to every parallel bivector.

If two bivectors, $\mu + \iota\nu$, $\mu' + \iota\nu'$, which do not lie in the same plane are perpendicular, we may resolve μ and ν into components parallel and perpendicular to the plane of μ' and ν'. The components perpendicular to the plane evidently contribute nothing to the value of

$$[\mu + \iota\nu].[\mu' + \iota\nu'].$$

Therefore the components of μ and ν parallel to the plane of μ', ν', form a bivector which is perpendicular to $\mu' + \iota\nu'$. That is, if two bivectors are perpendicular, the directional ellipse of either, projected upon the plane of the other and rotated through a quadrant in that plane, will be similar and similarly situated to the directional ellipse of the second.

8. A bivector may be divided in one and only one way into parts parallel and perpendicular to another, provided that the second is not circular. If $\mathfrak{a}$ and $\mathfrak{b}$ are the bivectors, the parts of $\mathfrak{a}$ will be

$$\frac{\mathfrak{b} \, . \, \mathfrak{a}}{\mathfrak{b} \, . \, \mathfrak{b}} \mathfrak{b} \quad \text{and} \quad \mathfrak{a} - \frac{\mathfrak{b} \, . \, \mathfrak{a}}{\mathfrak{b} \, . \, \mathfrak{b}} \mathfrak{b} \, .$$

If $\mathfrak{b}$ is circular, the resolution of $\mathfrak{a}$ is impossible, unless it is perpendicular to $\mathfrak{b}$. In this case the resolution is indeterminate.

9. Since $\mathfrak{a} \times \mathfrak{b} \, . \, \mathfrak{a} = 0$, and $\mathfrak{a} \times \mathfrak{b} \, . \, \mathfrak{b} = 0$, $\mathfrak{a} \times \mathfrak{b}$ is perpendicular to $\mathfrak{a}$ and $\mathfrak{b}$. We may regard the plane of the product as determined by the condition that the directional ellipses of the factors projected upon it become similar and similarly situated. The directional ellipse of the product is similar to these projections, but its orientation is different by 90°. It may easily be shown that $\mathfrak{a} \times \mathfrak{b}$ vanishes only with $\mathfrak{a}$ or $\mathfrak{b}$, or when $\mathfrak{a}$ and $\mathfrak{b}$ are parallel.

10. The bivector equation

$$(\mathfrak{a} \times \mathfrak{b} \, . \, \mathfrak{c}) \mathfrak{d} - (\mathfrak{b} \, . \, \mathfrak{c} \times \mathfrak{d}) \mathfrak{a} + (\mathfrak{c} \, . \, \mathfrak{d} \times \mathfrak{a}) \mathfrak{b} - (\mathfrak{d} \, . \, \mathfrak{a} \times \mathfrak{b}) \mathfrak{c} = 0$$

is identical, as may be verified by substituting expressions of the form $xi + yj + zk$ (x, y, z being biscalars), for each of the bivectors. (Compare No. 37.) This equation shows that if the product $\mathfrak{a} \times \mathfrak{b}$ of any two bivectors vanishes, one of these will be equal to the other with a biscalar coefficient, that is, they will be parallel, according to the definition given above. If the product $\mathfrak{a} \, . \, \mathfrak{b} \times \mathfrak{c}$ of any three bivectors vanishes, the equation shows that one of these may be expressed as a sum of the other two with biscalar coefficients. In this case, we may say (from the analogy of the scalar analysis) that the three bivectors are complanar. (This does not imply that they lie in any same real plane.) If $\mathfrak{a} \, . \, \mathfrak{b} \times \mathfrak{c}$ is not equal to zero, the equation shows that any fourth bivector may be expressed as a sum of $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{c}$ with biscalar coefficients, and indicates how these coefficients may be determined.

11. The equation

$$(\mathfrak{r} \, . \, \mathfrak{a}) \mathfrak{b} \times \mathfrak{c} + (\mathfrak{r} \, . \, \mathfrak{b}) \mathfrak{c} \times \mathfrak{a} + (\mathfrak{r} \, . \, \mathfrak{c}) \mathfrak{a} \times \mathfrak{b} = (\mathfrak{a} \times \mathfrak{b} \, . \, \mathfrak{c}) \mathfrak{r}$$

is also identical, as may easily be verified. If we set

$$\mathfrak{c} = \mathfrak{a} \times \mathfrak{b} \, ,$$

and suppose that

$$\mathfrak{r} \, . \, \mathfrak{a} = 0, \quad \mathfrak{r} \, . \, \mathfrak{b} = 0,$$

the equation becomes

$$(\mathfrak{r} \, . \, \mathfrak{a} \times \mathfrak{b}) \mathfrak{a} \times \mathfrak{b} = (\mathfrak{a} \times \mathfrak{b} \, . \, \mathfrak{a} \times \mathfrak{b}) \mathfrak{r} \, .$$

This shows that if a bivector $\mathfrak{r}$ is perpendicular to two bivectors $\mathfrak{a}$ and $\mathfrak{b}$, which are not parallel, $\mathfrak{r}$ will be parallel to $\mathfrak{a} \times \mathfrak{b}$. Therefore all bivectors which are perpendicular to two given bivectors are parallel to each other, unless the given two are parallel.

[*Note by Editors*.—The notation $|\Phi|\Phi_c^{-1}$, used in No. 140, was later improved by the author by the introduction of his Double Multiplication, according to which the above expression is represented by Φ_2, and $|\Phi|$ by Φ_3. For an extended treatment of Professor Gibbs's researches on Double Multiplication in their application in Vector Analysis see pp. 306-321, and 333 of "Vector Analysis," by E. B. Wilson, Chas. Scribner's Sons, New York, 1901.]