QUATERNIONS AND THEIR GENERALIZATIONS

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1. The discovery of quaternions by W. R. Hamilton in 1843 has led to an extensive theory of linear algebras (or closed systems of hypercomplex numbers) in which the quaternion algebra plays an important rôle. Frobenius[1] proved that the only real linear associative algebras in which a product is zero only when one factor is zero are the real number system, the ordinary complex number system, and the algebra of real quatermons. A much simpler proof has been given by the writer.[2] Later, the writer[3] was led to quaternions very naturally by means of the four-parameter continuous group which leaves unaltered each line of a set of rulings on the quadric surface $x_1^2+x_2^2+x_3^2+x_4^2=0$.

The object of the present note is to derive the algebra of quaternions and its direct generalizations without assuming the associative or commutative law. I shall obtain this interesting result by two distinct methods.

2. The term *field* will be employed here to designate any set of ordinary complex numbers which is closed under addition, subtraction, multiplication, and division. Thus all complex numbers form a field, likewise all real numbers, or all rational numbers.

Just as a couple (a,b) of real numbers defines an ordinary complex number $a+bi$, where $i^2=-1$, so also an n-tuple $(x_1,...,x_n)$ of numbers of a field F defines a hypercomplex number

$$x = x_1e_1+x_2e_2+...+x_ne_n, \tag{1}$$

where the *units* e_1, ..., e_n are linearly independent with respect to the field F and possess a multiplication table

$$e_ie_j=\sum_{k=1}^{n}\gamma_{ijk}e_k \qquad (i,j=1,...,n), \tag{2}$$

in which the γ's are numbers of F. Let $x'=\sum x_ie_i$ be another hypercomplex number

[1]Frobenius, *Jour. für Math.*, **84**, 1878 (59).
[2]Dickson, *Linear Algebras*, "Cambridge Tracts in Mathematics and Mathematical Physics," No. **16**, 1914 (10-12).
[3]Dickson, *Bull. Amer. Math. Soc.*, **22**, 1915 (53-61).

whose coördinates x'_i are numbers of F. Then shall

$$xx' = \sum_{i,j=1}^{n} x_i x'_j . e_i e_j, \quad x \pm x' = \sum_{i=1}^{n} (x_i \pm x'_i) e_i, \quad fx = xf = \sum_{i=1}^{n} f x_i . e_i,$$

when f is in F, so that multiplication is distributive. Under these assumptions, the set of all numbers (1) with coördinates in F shall be called a *linear algebra* over F.

3. We assume that e_1 is a principal unit (modulus), so that $e_1 x = x e_1 = x$ for every number x of the algebra, and write 1 for e_1. We assume that every number of the algebra satisfies a quadratic equation with coefficients in F. If $e^2 + 2ae + b = 0$, $(e+a)^2 = a^2 - b$, so that we may take the units to be 1, E_2, ..., E_n, where $E_i^2 = s_{ii}$, a number of F. For i and j distinct and > 1, $E_i \pm E_j$ satisfies a quadratic, so that $(E_i \pm E_j)^2 = s_{ii} + s_{jj} \pm (E_i E_j + E_j E_i)$ is a linear function of $E_i \pm E_j$. Thus $E_i E_j + E_j E_i$ is a linear function of $E_i + E_j$ and of $E_i - E_j$ and hence is a number $2s_{ij} = 2s_{ji}$ of F.

Let u_2, ..., u_n be arbitrary numbers of F and write $U = \sum u_k E_k$. Then U^2 equals

$$Q = \sum_{k,l=2}^{n} s_{kl} u_k n_l.$$

It is a standard theorem that Q can be reduced to $\sum c_i v_i^2$ by a linear transformation $u_k = \sum a_{kl} v_l$ with coefficients in F and of determinant $\neq 0$. Write

$$e_l = \sum_{k,l=2}^{n} a_{kl} E_k \qquad (l = 2, ..., n).$$

Then 1, e_2, ..., e_n are linearly independent and may be taken as the new units of our algebra over F. Then

$$U = \sum_{k,l} a_{kl} v_l E_k = \sum_{l} v_l e_l,$$
$$U^2 = \sum_{k,l} v_k v_l e_k e_l = \sum c_i v_i^2.$$

Hence

$$e_i^2 = c_i, \qquad e_i e_j + e_j e_i = 0 \qquad (i, j = 2, ..., n, \, i \neq j). \tag{3}$$

4. Write $x = x_1 + \sum x_i e_i$. Then $(x - x_1)^2 = \sum c_i x_i^2$. This gives $xx' = x'x = \sigma$, where

$$x' = 2x_1 - x = x_1 - \sum_{i=2}^{n} x_i e_i, \qquad \sigma = x_1^2 - \sum_{i=2}^{n} c_i x_i^2.$$

We shall call x' the *conjugate* to x and $\sigma = \sigma(x)$ the *norm* of x. Hence the product of any number and its conjugate in either order equals its norm. We assume that the norm of a product equals the product of the norms of the factors:

$$\sigma(x)\sigma(\xi) = \sigma(X), \qquad \text{if } x\xi = X, \tag{4}$$

and shall investigate the resulting types of linear algebras. We assume also that each $c_i \neq 0$ in (3).

5. By (2) the coördinates of $X = x\xi$ are $X_k = \sum x_i \xi_j \gamma_{ijk}$. Since $e_i^2 = c_i$, we have $\gamma_{ii1} = c_i$, $\gamma_{iik} = 0$ $(i > 1, k > 1)$. Hence

$$\begin{aligned} X_1 &= x_1\xi_1 + \sum_{i=2}^{n} x_i\xi_i c_i + \sum_{\substack{i,j=2,\\ i\neq j}}^{n} x_i\xi_j\gamma_{ij1}, \\ X_k &= x_1\xi_k + x_k\xi_1 + \sum_{\substack{i,j=2\\ i\neq j}}^{n} x_i\xi_j\gamma_{ijk} \qquad (k > 1). \end{aligned} \tag{5}$$

Since this transformation is the identity $X = x$ if $\xi = 1$, we obtain an infinitesimal transformation by taking $\xi_1 = 1$, $\xi_j = \delta t$, $\xi_i = 0$ $(i \neq 1, j)$:

$$\begin{aligned} \delta x_1 = X_1 - x_1 &= \left\{ c_j x_j + \sum_{\substack{i=2\\ i\neq j}}^{n} \gamma_{ij1} x_i \right\} \delta t, \\ \delta x_j &= \left\{ x_1 + \sum_{\substack{i=2\\ i\neq j}}^{n} y_{ijj} x_i \right\} \delta t, \\ \delta x_k &= \left\{ \sum_{\substack{i=2\\ i\neq j}}^{n} \gamma_{ijk} x_i \right\} \delta t \qquad (k \neq 1, j). \end{aligned} \tag{6}$$

For these ξ's, $\sigma(\xi)$ is unity to within an infinitesimal of the second order. Hence the increment to $\sigma(x)$ must vanish identically, so that

$$\gamma_{ij1} = \gamma_{ijj} = \gamma_{iji} = 0 \qquad (1,\ i,\ j \text{ distinct}), \tag{7}$$

$$c_k\gamma_{ijk} + c_i\gamma_{kji} = 0 \qquad (1,\ i,\ j,\ k \text{ distinct}). \tag{8}$$

By (7), (5) simplifies to

$$X_1 = x_1\xi_1 + \sum_{i=2}^{n} x_i\xi_i c_i, \quad X_k = x_1\xi_k + x_k\xi_1 + \sum x_i\xi_j\gamma_{ijk} \quad (k > 1), \tag{9}$$

where, in the final sum, i and j range over distinct values from 2, ..., n excluding k. This final sum is, therefore, absent if[4] $n = 3$; whence $\sigma(X)$ has the term $2x_2\xi_2c_2.x_3\xi_3c_3$ which does not occur in $\sigma(x)\sigma(\xi)$. But $c_2c_3 \neq 0$ by hypothesis. Hence $n > 3$.

Hitherto we have not examined the conditions which follow from the final equations (3); these are

$$\gamma_{jik} = -\gamma_{ijk} \qquad (i, j = 2, ..., n;\ i \neq j). \tag{10}$$

[4] By (7), $e_2e_3 = e_3e_2 = 0$.

6. Taking $n=4$ and applying (10), we see that (9) become

$$\begin{cases} X_1 &= x_1\xi_1 + c_2x_2\xi_2 + c_3x_3\xi_3 + c_4x_4\xi_4, \\ X_2 &= x_1\xi_2 + x_2\xi_1 + \gamma_{342}(x_3\xi_4 - x_4\xi_3), \\ X_3 &= x_1\xi_3 + x_3\xi_1 + \gamma_{243}(x_1\xi_4 - x_4\xi_1), \\ X_4 &= x_1\xi_4 + x_4\xi_1 + \gamma_{234}(x_2\xi_3 - x_3\xi_2). \end{cases} \tag{11}$$

These transformations do not in general form a group and hence are not generated by the corresponding infinitesimal transformations employed above. Hence it remains to require that $\sigma(X) = \sigma(x)\sigma(\xi)$ under the transformations (11). The conditions are seen to be

$$c_3c_4 = -c_2\gamma_{342}^2, \quad c_2c_4 = -c_3\gamma_{243}^2, \quad c_2c_3 = -c_4\gamma_{234}^2, \quad c_4\gamma_{234} = c_2\gamma_{342} = -c_3\gamma_{243},$$

the first two of which reduce to the third by means of the last three equations. To these last can be reduced all the conditions (8) by means of (10).

Applying the transformation of variables which multiplies x_4, ξ_4, X_4 by γ_{234}, and leaves the remaining x_i, ξ_i, X_i unaltered, we get

$$\begin{cases} X_1 &= x_1\xi_1 + c_2x_2\xi_2 + c_3x_3\xi_3 - c_2c_3x_4\xi_4, \\ X_2 &= x_1\xi_2 + x_2\xi_1 - c_3x_3\xi_4 + c_3x_4\xi_3, \\ X_3 &= x_1\xi_3 + x_3\xi_1 + c_2x_2\xi_4 - c_2x_4\xi_2, \\ X_4 &= x_1\xi_4 + x_4\xi_1 + x_2\xi_3 - x_3\xi_2. \end{cases} \tag{11$'$}$$

These are the values obtained by Lagrange[5] in his generalization $\sigma(x)\sigma(\xi) = \sigma(X)$ of Euler's formula for the product of two sums of four squares. Then $x\xi = X$ gives the following multiplication table for the units:

$$\begin{cases} e_2^2 = c_2 \quad e_3^2 = c_3, \quad e_4^2 = -c_2c_3, \quad e_2e_3 = e_4, \quad e_3e_2 = -e_4, \\ e_2e_4 = c_2e_3, \quad e_4e_2 = -c_2e_3, \quad e_3e_4 = -c_3e_2, \quad e_4e_3 = c_3e_2. \end{cases} \tag{12}$$

This algebra is associative and is the direct generalization of quaternions to a general field F which the writer[6] obtained elsewhere from assumptions including associativity. The four-rowed determinants of the general number x of this algebra equals $\sigma^2(x)$. The case $c_2 = c_3 = -1$ gives the algebra of quaternions, for which it is customary to write i, j, k instead of our units e_2, e_3, e_4.

7. It is not very laborious to show by the above method that the cases $n=5$ and $n=6$ are excluded. However, Hurwitz has proved that a relation of the form $\sigma(x)\sigma(\xi) = \sigma(X)$ is impossible if $n \neq 1,2,4,8$. A slight simplification of his proof, together with an account of the history of this problem, has been given by the writer.[7] Hurwitz made no attempt to find all solutions when $n=4$. We proceed to treat this problem.

[5]Lagrange, *Nouv. Mém. Acad. Roy. Sc. de Berlin*, année 1770, Berlin, 1772 (123-133); *Oeuvres* de Lagrange, **3**, 1869 (189). Reproduced in Dickson's *History of the Theory of Numbers*, II, 1920 (279-281).

[6]Dickson, *Trans. Amer. Math. Soc.*, **13**, 1912 (65).

[7]Dickson, *Annals of Math.*, **20**, 1919 (155-171, 297).

Consider the case $c_i = -1$ to which the general case may be reduced by an irrational transformation. Then $\sigma(x) = \sum x_i^2$. We investigate the linear algebras having property (4), i. e.,

$$(x_1^2 + ... + x_n^2)(\xi_1^2 + ... + \xi_n^2) = X_1^2 + ... + X_n^2, \tag{13}$$

if

$$X_k = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \gamma_{ijk} x_i \right) \xi_j \qquad (k = 1, ..., n). \tag{14}$$

The matrix M of this substitution has the element $\sum_{i=1}^{i=n} \gamma_{ijk} x_i$ in the kth row and jth column. If this substitution is applied to a quadratic form in X_1, ..., X_n of matrix Q, it is a standard theorem that we obtain a quadratic form in ξ_1, ..., ξ_n, whose matrix is $M'QM$, where M' is the transposed of M, being obtained from M by the interchange of its rows and columns. In our problem, Q is the identity matrix I whose elements are all zero except the diagonal elements which are 1. Hence, by (13),

$$M'M = (x_1^2 + ... + x_n^2)I. \tag{15}$$

When a homogeneous polynomial $\sigma(x_1, ..., x_n)$ of any degree has the property (4) of possessing a theorem of multiplication, the writer[8] has proved that we may apply a linear transformation on x_1, ..., x_n which leave $\sigma(x)$ unaltered and one on ξ_1, ..., ξ_n which leaves $\sigma(\xi)$ unaltered such that the new algebra has the principal unit e_1, so that γ_{1jk} and γ_{j1k} are both 0 if $j \neq k$, and both unity if $j = k$.

Hence $M = x_1 M_1 + ... + x_n M_n$, where γ_{ijk} is the element in the kth row and jth column of M_i, whence $M_i = I$. Thus (15) gives

$$M_i' = -M_i, \quad M_i' M_i = I, \quad M_i' M_j + M_j' M_i = 0 \quad (i > 1, j > 1, j \neq i). \tag{16}$$

In view of the values of γ_{jik}, and $M_i' = -M_i$, we have, when $n = 4$,

$$M_2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -\gamma_{223} & -\gamma_{224} \\ 0 & \gamma_{223} & 0 & -\gamma_{234} \\ 0 & \gamma_{224} & \gamma_{234} & 0 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & -\gamma_{323} & -\gamma_{324} \\ 1 & \gamma_{323} & 0 & -\gamma_{334} \\ 0 & \gamma_{324} & \gamma_{334} & 0 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -\gamma_{423} & -\gamma_{424} \\ 0 & \gamma_{423} & 0 & -\gamma_{434} \\ 1 & \gamma_{424} & \gamma_{434} & 0 \end{pmatrix}.$$

[8] Dickson, *Comptes Rendus du Congrès Internat. Math.*, Strasbourg, 1920 (131-146).

By $M_i'M_i = I$, we have $M^{2^i} = -I$, which gives

$$\gamma_{223} = \gamma_{224} = \gamma_{323} = \gamma_{334} = \gamma_{424} = \gamma_{434} = 0, \qquad \gamma^2 = \delta^2 = \varepsilon^2 = 1,$$

where $\gamma = \gamma_{234}$, $\delta = \gamma_{324}$, $\varepsilon = \gamma_{423}$. Hence

$$M_2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\gamma \\ 0 & 0 & \gamma & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\delta \\ 1 & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -\varepsilon & 0 \\ 0 & \varepsilon & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{17}$$

The final condition (16) states that M_iM_j is skew-symmetric. The products M_2M_3, M_2M_4, M_3M_4 of matrices (17) are seen at once to be skew-symmetric if and only if $\delta = -\gamma$, $\varepsilon = \gamma$, and then $M_2M_3 = \gamma M_4$, $M_2M_4 = -\gamma M_3$, $M_3M_4 = \gamma M_2$. Writing i, j, k for M_2, M_3, γM_4, we have the multiplication table of quaternions. Or we may form the matrix M and write X_k for the sum of the products of the elements of its kth row by ξ_1, ..., ξ_4, and take $\gamma = 1$ (by multiplying x_4, ξ_4, X_4 by γ); we obtain $(11')$ for $c_2 = c_3 = -1$. Hence we have again obtained the quaternion algebra without assuming the associative law. The case $n = 8$ is being investigated in this way by one of my students.