On the Function arc sin (x+iy).

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The determination of the function in question, the arc to a given imaginary sine, is considered in Cauchy's *Exercises d'Analyse, &c.*, t. III. (1844), p. 382; but it appears, by two hydrodynamical papers by Mr. Ferrers and Mr. Lamb, *Quarterly Mathematical Journal*, t. XIII. (1874), p. 115, and t. XIV. (1875), p. 40, that the question is connected with the theory of confocal conics.

Taking $c = \sqrt{a^2 - b^2}$ a positive real quantity which may ultimately be put $= 1$, the question is to find the real quantities ξ, η, such that

$$\xi + i\eta = \arcsin \frac{1}{c}(x + iy),$$

or say

$$x + iy = c\sin(\xi + i\eta),$$

so that

$$x = c\sin\xi\cos i\eta, \qquad iy = c\cos\xi\sin i\eta.$$

It is convenient to remark that if a value of $\xi + i\eta$ be $\xi' + i\eta'$, then the general value is $2m\pi + \xi' + i\eta'$ or $(2m+1)\pi - (\xi' + i\eta')$; hence, η may be made positive or negative at pleasure; $\cos i\eta$ is in each case positive, but $\frac{1}{i}\sin i\eta$ has the same sign as η; hence $\cos\xi$ has the same sign as x, but $\sin\xi$ has the same sign as y or the reverse sign, according as η is positive or negative; for any given values of x and y, we obtain, as will appear, determinate positive values of $\sin^2\xi$ and $\cos^2\xi$; and the square roots of these must therefore be taken so as to give to $\sin\xi$, $\cos\xi$ their proper signs respectively.

Suppose that λ, μ are the elliptic coordinates of the point (x, y); viz. that we have

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$$
$$\frac{x^2}{a^2+\mu} + \frac{y^2}{b^2+\mu} = 1$$

where $a^2 + \lambda$, $b^2 + \lambda$, and $a^2 + \mu$ are positive, but $b^2 + \mu$ is negative. Calling ρ, σ the distances of the point x, y from the points $(c,0)$ and $(-c,)$, that is, assuming

$$\rho = \sqrt{(x-c)^2+y^2},$$
$$\sigma = \sqrt{(x+c)^2+y^2},$$

then we have

$$\begin{aligned}\sqrt{a^2+\lambda} &= \tfrac{1}{2}(\sigma+\rho),\\ \sqrt{a^2+\mu} &= \tfrac{1}{2}(\sigma-\rho),\end{aligned} \quad \text{whence also} \quad \begin{aligned}\sqrt{b^2+\lambda} &= \tfrac{1}{2}\sqrt{(\sigma+\rho)^2-4c^2},\\ \sqrt{b^2+\mu} &= \tfrac{1}{2}\sqrt{(\sigma-\rho)^2-4c^2},\end{aligned}$$

which equations determine λ, μ as functions of x, y.

Now we have

$$\begin{aligned}\rho\sigma &= \sqrt{(x^2+y^2-c^2)^2-4c^2x^2} = \sqrt{(x^2-y^2-c^2)^2+4x^2y^2},\\ \rho^2+\sigma^2 &= 2(x^2+y^2+c^2);\end{aligned}$$

substituting herein for x, y their values

$$c\sin\xi\cos i\eta, \quad -ci\cos\xi\sin i\eta,$$

we find

$$\begin{aligned}x^2-y^2-c^2 &= c^2\{\sin^2\xi\cos^2 i\eta+\cos^2\xi\sin^2 i\eta-(\sin^2\xi+\cos^2\xi)(\sin^2 i\eta+\cos^2 i\eta)\}\\ &= -c^2(\sin^2\xi\sin^2 i\eta+\cos^2\xi\cos^2 i\eta),\end{aligned}$$

whence

$$\begin{aligned}(x^2-y^2-c^2)^2+4x^2y^2 &= c^4(\cos^2\xi\cos^2 i\eta+\sin^2\xi\sin^2 i\eta)^2-4c^4\sin^2\xi\cos^2\xi\sin^2 i\eta\cos^2 i\eta\\ &= c^4(\cos^2\xi\cos^2 i\eta-\sin^2\xi\sin^2 i\eta)^2.\end{aligned}$$

Hence

$$2\rho\sigma = 2c^2(\cos^2\xi\cos^2 i\eta-\sin^2\xi\sin^2 i\eta),$$

and

$$\rho^2+\sigma^2 = 2c^2(\sin^2\xi\cos^2 i\eta-\cos^2\xi\sin^2 i\eta+1);$$

hence

$$\begin{aligned}(\rho+\sigma)^2 &= 2c^2(\cos^2 i\eta-\sin^2 i\eta+1), = 4c^2\cos^2 i\eta,\\ (\rho-\sigma)^2 &= 2c^2(\sin^2\xi-\cos^2\xi+1), = 4c^2\sin^2\xi.\end{aligned}$$

Consequently

$$\begin{aligned}a^2+\lambda &= c^2\cos^2 i\eta,\\ a^2+\mu &= c^2\sin^2\xi,\end{aligned} \quad \text{and thence} \quad \begin{aligned}b^2+\lambda &= -c^2\sin^2 i\eta,\\ b^2+\mu &= -c^2\cos^2\xi,\end{aligned}$$

values which verify as they should do the equations

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1,$$
$$\frac{x^2}{a^2+\mu}+\frac{y^2}{b^2+\mu}=1,$$

viz. these become

$$\frac{x^2}{c^2\cos^2 i\eta}+\frac{y^2}{-c^2\sin^2 i\eta}=\sin^2\xi+\cos^2\xi,=1,$$
$$\frac{x^2}{c^2\sin^2\xi}+\frac{y^2}{-c^2\cos^2\xi}=\cos^2 i\eta+\sin^2 i\eta,=1.$$

The same equations, or as we may also write them,

$$\lambda=-a^2\sin^2 i\eta-b^2\cos^2 i\eta,$$
$$\mu=-a^2\cos^2\xi-b^2\sin^2\xi,$$

determine η as a function of λ, and ξ as a function of μ; λ, μ being by what precedes, given functions of x, y.

Or more simply, starting from the last-mentioned values of λ, μ, and substituting these in the expressions

$$x^2=\frac{(a^2+\lambda)(a^2+\mu)}{a^2-b^2},\qquad y^2=\frac{(b^+\lambda)(b^2+\mu}{b^2-a^2},$$

we find

$$x^2=c^2\sin^2\xi\cos^2 i\eta,\qquad y^2=-c^2\cos^2\xi\sin^2 i\eta,$$

or say

$$x=c\sin\xi\cos i\eta,\qquad iy=c\cos\xi\sin i\eta,$$

whence

$$x+iy=c\sin(\xi+i\eta),$$

the original relation between x, y and ξ, η.