ON THE APPLICATION OF QUATERNIONS TO THE THEORY OF ROTATION.

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In a paper published in the *Philosophical Magazine*, February 1845, [20], I showed how some formulæ of M. Olinde Rodrigues relating to the rotation of a solid body might be expressed in a very simple form by means of Sir W. Hamilton's theory of quaternions. The property in question may be thus stated. Suppose a solid body which revolves through an angle θ round an axis passing through the origin and inclined to the axes of coordinates at angles a, b, c. Let

$$\lambda = \tan\tfrac{1}{2}\theta\cos a, \qquad \mu = \tan\tfrac{1}{2}\theta\cos b, \qquad \nu = \tan\tfrac{1}{2}\theta\cos c,$$

and write

$$\Lambda = 1 + i\lambda + j\mu + k\nu;$$

let x, y, z be the coordinates of a point in the body previous to the rotation, x_1, y_1, z_1 those of the same point after the rotation, and suppose

$$\begin{aligned}\Pi &= ix + jy + kz,\\ \Pi_1 &= ix_1 + jy_1 + kz_1;\end{aligned}$$

then the coordinates after the rotation may be determined by the formula

$$\Pi = \Lambda\Pi\Lambda^{-1};$$

viz., developing the second side of this equation,

$$\begin{aligned}\Pi_1 = {} & i(\alpha x + \beta y + \gamma z)\\ & + j(\alpha' x + \beta' y + \gamma' z)\\ & + k(\alpha'' x + \beta'' y + \gamma'' z),\end{aligned}$$

where, putting to abbreviate $\kappa = 1 + \lambda^2 + \mu^2 + \nu^2$, we have

$$\begin{array}{lll}
\kappa\alpha = 1 + \lambda^2 - \mu^2 - \nu^2, & \kappa\alpha' = 2(\lambda\mu + \nu), & \kappa\alpha'' = 2(\lambda\nu - \mu), \\
\kappa\beta = 2(\lambda\mu - \nu), & \kappa\beta' = 1 - \lambda^2 + \mu^2 - \nu^2, & \kappa\beta'' = 2(\mu\nu + \lambda), \\
\kappa\gamma = 2(\lambda\nu + \mu), & \kappa\gamma' = 2(\mu\nu - \lambda), & \kappa\gamma'' = 1 - \lambda^2 - \mu^2 + \nu^2;
\end{array}$$

these values satisfying identically the well-known system of equations connecting the quantities α, β, γ, α', β', γ', α'', β'', γ''.

The quantities a, b, c, θ being immediately known when λ, μ, ν are known, these last quantities completely determine the direction and magnitude of the rotation, and may therefore be termed the coordinates of the rotation; Λ will be the quaternion of the rotation. I propose here to develope a few of the consequences which may be deduced from the preceding formulæ.

Suppose, in the first place, $\Pi = \Lambda - 1$, then $\Pi_1 = \Lambda - 1$, which evidently implies that the point is on the axis of rotation. The equation $\Pi_1 = \Pi$ gives the identical equations

$$\begin{array}{llll}
\lambda(\alpha - 1) & +\mu\beta & +\nu\gamma & = 0, \\
\lambda\alpha' & +\mu(\beta' - 1) & +\nu\gamma' & = 0, \\
\lambda\alpha'' & +\mu\beta'' & +\nu(\gamma'' - 1) & = 0;
\end{array}$$

from which, by changing the signs of λ, μ, ν, we derive

$$\begin{array}{llll}
\lambda(\alpha - 1) & +\mu\alpha' & +\nu\alpha'' & = 0, \\
\lambda\beta & +\mu(\beta' - 1) & +\nu\beta'' & = 0, \\
\lambda\gamma & +\mu\gamma' & +\nu(\gamma'' - 1) & = 0.
\end{array}$$

Hence evidently, whatever be the value of Π,

$$\Lambda\Pi\Lambda^{-1} - \Pi = 0,$$

if after the multiplication i, j, k are changed into λ, μ, ν, a property which will be required in the sequel.

By changing the signs of λ, μ, ν, we also deduce

$$\begin{aligned}
\Lambda^{-1}\Pi\Lambda = {} & i(\alpha x + \alpha' y + \alpha'' z) \\
& + j(\beta x + \beta' y + \beta'' z) \\
& + k(\gamma x + \gamma' y + \gamma'' z),
\end{aligned}$$

where α, β, γ, α', β', γ', α'', β'', γ'' are the same as before.

Let the question be proposed to compound two rotations (both axes of rotation being supposed to pass through the origin). Let L be the first axis, Λ the quaternion of rotation, L' the second axis, which is supposed to be fixed in space, so as not to alter its direction by reason of the first rotation, Λ' the corresponding quaternion of rotation. The combined effect is given at once by

$$\Pi_1 = \Lambda' \left(\Lambda\Pi\Lambda^{-1}\right) \Lambda'^{-1},$$

that is,

$$\Pi_1 = \Lambda'\Lambda\Pi(\Lambda'\Lambda)^{-1};$$

or since (if Λ_1 be the quaternion for the combined rotation) $\Pi_1 = \Lambda_1\Pi\Lambda_1^{-1}$, we have clearly

$$\Lambda_1 = M_1\Lambda'\Lambda,$$

M_1 denoting the reciprocal of the real part of $\Lambda'\Lambda$, so that

$$M_1^{-1} = 1 - \lambda\lambda' - \mu\mu' - \nu\nu'.$$

Retaining this value, the coefficients of the combined rotation are given by

$$\begin{aligned}
\lambda_1 &= M_1(\lambda + \lambda' + \mu'\nu - \mu\nu'),\\
\mu_1 &= M_1(\mu + \mu' + \nu'\lambda - \nu\lambda'),\\
\nu_1 &= M_1(\nu + \nu' + \lambda'\mu - \lambda\mu');
\end{aligned}$$

to which may be joined [if $\kappa_1 = 1 + \lambda_1^2 + \mu_1^2 + \nu_1^2$],

$$\kappa_1 = M_1^2\kappa\kappa',$$

κ, κ', κ_1 as before. Λ or Λ' may be determined with equal facility in terms of Λ', Λ_1, or Λ, Λ_1. These formulæ are given in my paper on the rotation of a solid body (*Cambridge Mathematical Journal*, vol. III. p. 226, [6]).

If the axis L' be fixed in the body and moveable with it, its position after the first rotation is obtained from the formula $\Pi_1 = \Lambda\Pi\Lambda^{-1}$ by writing $\Pi = \Lambda' - 1$. Representing by $\Lambda'' - 1$ the corresponding value of Π_1, we have $\Lambda'' = \Lambda\Lambda'\Lambda^{-1}$, which is the value to be used instead of Λ' in the preceding formula for the combined rotation, thus the quaternion of rotation is proportional to $\Lambda\Lambda'\Lambda^{-1}\Lambda$, that is to $\Lambda\Lambda'$. Hence here

$$\Lambda_1 = M_1\Lambda\Lambda',$$

which only differs from the preceding in the order of the quaternion factors. If the fixed and moveable axes be mixed together in any order whatever, the fixed axes taken in order being $L, L', \ldots$ and the moveable axes taken in order being $L_0, L_0' \ldots$ then the combined effect of the rotations is given by

$$\Lambda_1 = M \ldots \Lambda''\Lambda'\Lambda\Lambda_0\Lambda_0' \ldots,$$

M being the reciprocal of the real term of the product of all the quaternions.

Suppose next the axes do not pass through the same point. If α, β, γ be the coordinates of a point in L, and

$$\Gamma = \alpha i + \beta j + \gamma k,$$

then the formula for the rotation is

$$\Pi_1 - \Gamma = \Lambda(\Pi - \Gamma)\Lambda^{-1},$$

or

$$\Pi_1 = \Lambda\Pi\Lambda^{-1} - (\Lambda\Gamma\Lambda^{-1} - \Gamma),$$

where the first term indicates a rotation round a parallel axis through the origin, and the second term a translation.

For two axes L, L' fixed in space,

$$\Pi_1 = \Lambda'\Lambda\Pi(\Lambda'\Lambda)^{-1} - (\Lambda'\Gamma'\Lambda'^{-1} - \Gamma') - \Lambda'(\Lambda\Gamma\Lambda^{-1} - \Gamma)\Lambda'^{-1};$$

and so on for any number, the last terms being always a translation. If the two axes are parallel, and the rotations equal and opposite,

$$\Lambda = \Lambda'^{-1},$$

whence

$$\Pi_1 = \Pi + \Lambda'(\Gamma - \Gamma')\Lambda'^{-1}(\Gamma - \Gamma');$$

or there is only a translation. The constant term vanishes if i, j, k are changed into λ', μ', ν', which proves that the translation is in a plane perpendicular to the axes.

Any motion of a solid body being represented by a rotation and a translation, it may be required to resolve this into two rotations. We have

$$\Pi_1 = \Lambda\Pi\Lambda_1^{-1} + T,$$

where T is a given quaternion whose constant term vanishes. Hence, comparing this with the general formula just given for the combination of two rotations,

$$\begin{aligned} \Lambda_1 &= M_1\Lambda'\Lambda, \\ T &= -(\Lambda'\Gamma'\Lambda'^{-1} - \Gamma') - \Lambda'(\Lambda\Gamma\Lambda^{-1} - \Gamma)\Lambda'^{-1}, \end{aligned}$$

the second of which equations may be simplified by putting $\Lambda'^{-1}T\Lambda' = S$, by which it may be reduced to

$$S = (\Lambda'^{-1}\Gamma'\Lambda' - \Gamma') - (\Lambda\Gamma\Lambda^{-1} - \Gamma),$$

which, with the preceding equation $\Lambda_1 = M_1\Lambda'\Lambda$, contains the solution of the problem. Thus if Λ or Λ' be given, the other is immediately known; hence also S is known. If in the last equation, after the multiplication is completely effected, we change i, j, k into λ, μ, ν, or λ', μ', ν', we have respectively,

$$S = \Lambda'^{-1}\Gamma'\Lambda' - \Gamma', \qquad S = -(\Lambda\Gamma\Lambda^{-1} - \Gamma),$$

which are equations which must be satisfied by the coefficients of Γ' and Γ respectively. Thus if the direction of one axis is given, that of the other is known, and the axes must lie in certain known planes. If the position of one of the axes in its plane be assumed, the equation containing S divides itself into three others (equivalent to two independent equations) for the determination of the position in its plane of the other axis. If the axes are parallel, λ, μ, ν are proportional to λ', μ', ν'; or changing i, j, k into λ, μ, ν, or λ', μ', ν', we have $S = 0$; or what is the same thing, $T = 0$, which shows that the translation must be perpendicular to the plane of the two axes.

If p, q, r have their ordinary signification in the theory of rotation, then from the values in the paper in the *Cambridge Mathematical Journal* already quoted,

$$\kappa(ip + jq + kr) = 2\frac{d\Lambda}{dt}\Lambda + \frac{d\kappa}{dt};$$

but I have not ascertained whether this formula leads to any results of importance. It may, however, be made use of to deduce the following property of quaternions, viz. if $\Lambda_1 = M_1\Lambda'\Lambda$, M_1 as before, then

$$\frac{1}{\kappa_1}\left(2\frac{d\Lambda_1}{dt}\Lambda_1 + \frac{d\kappa_1}{dt}\right) = \frac{1}{\kappa}\left(2\frac{d\Lambda}{dt}\Lambda + \frac{d\kappa}{dt}\right),$$

in which the coefficients of Λ' are considered constant.

To verify this *à posteriori*, if in the first place we substitute for κ_1 its value $M_1^2\kappa\kappa'$, we have

$$\frac{d\kappa_1}{dt} = M_1^2\kappa'\frac{d\kappa}{dt} + \frac{2}{M_1}\frac{dM_1}{dt}\kappa_1,$$

and thence

$$\frac{d\Lambda_1}{dt}\Lambda_1 + \frac{1}{M_1}\frac{dM_1}{dt}\kappa_1 = M^2\kappa'\frac{d\Lambda}{dt}\Lambda.$$

Also

$$\frac{d\Lambda_1}{dt}\Lambda_1 = \left(\frac{1}{M_1}\frac{dM_1}{dt}\Lambda_1 + M_1\Lambda'\frac{d\Lambda}{dt}\right)\Lambda_1 = \frac{1}{M_1}\frac{dM_1}{dt}\Lambda_1^2 + M_1^2\Lambda'\frac{d\Lambda}{dt}\Lambda'\Lambda,$$

which reduces the equation to

$$\frac{1}{M_1}\frac{dM_1}{dt}(\Lambda_1^2+\kappa_1)+M_1^2\Lambda'\frac{d\Lambda}{dt}\Lambda'\Lambda = M_1^2\kappa_1\frac{d\Lambda}{dt}\Lambda.$$

Hence observing that

$$\Lambda_1^2+\kappa_1 = 2\Lambda_1 = 2M_1\Lambda'\Lambda,$$

and omitting the factor Λ from the resulting equation,

$$\frac{2}{M_1^2}\frac{dM_1}{dt}\Lambda'+\Lambda'\frac{d\Lambda}{dt}\Lambda' = \kappa'\frac{d\Lambda}{dt};$$

or since

$$\frac{1}{M_1} = 1-\lambda\lambda'-\mu\mu'-\nu\nu',$$

substituting and dividing by Λ', we obtain

$$2\left(\lambda'\frac{d\lambda}{dt}+\mu'\frac{d\mu}{dt}+\nu'\frac{d\nu}{dt}\right) = \kappa'\Lambda'^{-1}\frac{d\Lambda}{dt}-\frac{d\Lambda}{dt}\Lambda',$$

or finally,

$$\begin{aligned}2\left(\lambda'\frac{d\lambda}{dt}+\mu'\frac{d\mu}{dt}+\nu'\frac{d\nu}{dt}\right) =&(1-i\lambda'-j\mu'-k\nu')\left(i\frac{d\lambda}{dt}+j\frac{d\mu}{dt}+k\frac{d\nu}{dt}\right)\\ &-\left(i\frac{d\lambda}{dt}+j\frac{d\mu}{dt}+k\frac{d\nu}{dt}\right)(1+i\lambda'+j\mu'+k\nu')\end{aligned}$$

$$=-(i\lambda'+j\mu'+k\nu')\left(i\frac{d\lambda}{dt}+j\frac{d\mu}{dt}+k\frac{d\nu}{dt}\right)-\left(i\frac{d\lambda}{dt}+j\frac{d\mu}{dt}+k\frac{d\nu}{dt}\right)(i\lambda'+j\mu'+k\nu'),$$

which is obviously true.

C.