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Cantor and generalized continuum hypotheses may be false

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Abstract

Cantor and generalized contunuum hypotheses are shown to be possibly factually false. Two new hyper-continuum hypotheses are posted to challenge pure mathematics and stir investigations of the abstract multispatial hyperspace, which is important for physical applications. Unlike the Cantorian hypotheses, however, the new ones conjecture existence of certain distinct, mixed kind of hyper-continuum that may serve as the manifold over which dual pairs of linear vector spaces with quite unrestricted abstract operations could be effectively constructed.

1. New crises in the old pure mathematics

Cardinal number $\tilde{\tilde{A}}$ of a set A is the class of all sets equipollent to A [1]. Georg Cantor used double bar to indicate two levels of abstraction. I will use wavy bars to snub poorly handling of this concept in pure mathematics (PM). I am not trying to bash Georg Cantor, but the postulative, art-like PM tolerated key theoretical mistakes for many millennia [2] or centuries [3]. It virtually censored critique of its paradoxes and inconsistencies, which made some applications nearly impossible. I am writing this paper for those who are adversely affected by the PM, which defiantly rejected corrections and deflected almost every criticism of its often self-inflicted ''wounds''.

The double abstraction was with respect to the nature and the order of the elements of sets [4]. Natural numbers are linearly ordered and thus one can always designate the next natural number, whose set was sometimes called 'upward directed' [5]. Rational numbers could be ordered in several ways and because of this feature one cannot designate the next rational number. This fact makes little qualitative difference between the natural numbers—or integers, in general—and the rational numbers. Cantorian set theory (ST) neglected the difference, because the essence of cardinality is the feasibility of counting numbers—or enumerate them when dealing with infinite sets. Irrational and consequently thus also real numbers are not denumerable, however [6]. They do form dense multitude called continuum. Even though arithmetic operations are performed on single numbers, the ST left the operational side to algebras and investigated only properties of their sets.

PM invented abstract algebraic structures such as groups, rings, ideals and fields [7–9]. Groups are sets with an operation defined on them. Fields featured all four arithmetic operations except for division by zero. Number fields may correspond to scalar physical fields. Since nature was never troubled by prohibition on division by zero, I removed it [2]. I introduced fully operational bigroup, which is additive and multiplicative group at once [2]. Although ring is set with both addition and multiplication [10,11]. I envisioned bigroup as an operational structure, a set whose operands may contain some elements other than numbers [2]. Even though infinity is not a number [12], it can be an operand [2] just as vector is. By dealing with operations only in terms of sets, PM's algebras placed artificial operational restrictions on

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algebraic and geometric structures. This feature tormented even great mathematicians by making their inventions somewhat evasive [13–17]. Although former algebraic approach to mappings of sets by means of morphisms is important, it often mixed up abstract operations and sets.

Former algebraic approach was formally perfect, but it merely postulated existence of sets and morphisms [18] without showing methods to construct them. The primary concern of modern algebras is not how an operation can be performed, but whether it maps *into* or *onto* and the like abstract issues [19–23]. As important as this may be for proofs, the nature does not really care about all that. The PM's concerns were not constructive, even though theoretically significant. We need thus an approach that is more relevant to operations performed in nature, which never complained about morphisms or the allegedly impossible division by zero, as far as I can tell. Abstract sets and morphisms should be de-emphasized as hardly operational. My decision to come up with a definite way to implement the feared division by zero was not really arbitrary, however. It has removed a hidden paradox from number theory and an obvious absurd from algebraic group theory. It was necessary step for full deployment of constructive, synthetic mathematics (SM) [2,3].

Problems hidden in PM implicitly affect all who use mathematics, even though we may not always be aware of their adverse impact on our thinking. Just take a look at the paradox that emerges from the usual prescription for multiplication of zeros that remained uncontested for some 5000 years

$$0 \cdot 0 = 0 \Rightarrow 0 \cdot 1/\infty = 0 \Rightarrow 0 \cdot 1 = 0 \cdot \infty \Rightarrow 1(? = ?)\infty \tag{0a}$$

This "fact" was covered up by the infamous prohibition on division by zero [2]. How ingenious. If one is prohibited from dividing by zero one could not obtain this paradox. Yet the prohibition did not really make anything right. It silenced objections to irresponsible reasonings and prevented corrections to the PM's flamboyant axiomatizations. The prohibition on treating infinity as invertible counterpart to zero did not do any good either. We use infinity in calculus for symbolic calculations of limits [24], for zero is the infinity's twin [25], and also in projective geometry as well as in geometric mapping of complex numbers. Therein a sphere is cast onto the plane that is tangent to it and its free (opposite) pole in a point at infinity [26–28]. Yet infinity as an inverse to the natural zero removes the whole absurd (0a), for we obtain [2]

$$0 = 1/\infty \Rightarrow 0 \cdot 0 = 1/\infty^2 > 0 \approx 0 \tag{0b}$$

Stereographic projection of complex numbers tacitly contradicted the PM's prescribed way to multiply zeros, yet it was never openly challenged. The old formula for multiplication of zeros (0a) is valid only as a practical approximation, but it is group-theoretically inadmissible in no-nonsense reasonings. The tiny distinction in formula (0b) makes profound theoretical difference for geometries and consequently also for physical applications. This is not about multiplying zeros, which I could not care less in practice, but it is about the conceptual, mathematical infrastructure of physics, where we just cannot afford making logically irresponsible shortcuts.

The nature does not really operate on numbers of her choice—it operates on operands, no matter what. If it takes an infinity and division by zero to make all abstract operations complete, so be it. Infinity could be introduced axiomatically [29,30]. Even logic contains veiled references to infinity [31]. Medieval thinkers considered infinity self-contradictory notion, until Galilei showed that infinite set can be put into 1-to-1 correspondence with a proper subset of itself [32]. Although Dedekind adopted this ingenious definition of infinity into the ST [33,34], it may cause severe conceptual problems if the infinity is not quantified [2]. The definition is very suitable for processes [35] and thus some authors prefer to define infinite set as not finite [36,37]. Yet infinity is well entrenched part of our unconscious conceptual system [38]. As an inverse to zero, the operational infinity could be understood in terms of imaginary unit, which acts as an inversion operator in algebras and geometries [3]. Instead of rejecting infinity as not a number, even though it annihilates PM's hidden paradoxes, we should perhaps ask ourselves what should the mathematical reality look like, since it treats the infinity as valid operand. Since nature handles infinity graciously, then we would not really be investigating the nature, if we reject the infinity. One may not like all the ways the nature works, but we should comply with her modes of operations, if we want to stay in touch with the physical reality, or nature, in short.

Georg Cantor originally investigated only numbers [39], but then in much more general approach he explicitly talks about set points in a space [40,41], identifying thus real numbers with their possible geometric representations even on a sphere [42]. He distinguished sparse point sets of isolated points [43] from intrinsically dense point sets, which he defined topologically [44]. He was primarily concerned with relationships between sets and its subsets [45] and declared that linear dimensions are irrelevant to the cardinality of sets [46]. This underscored qualitative character of the concept. He always emphasized in his papers two roles played by mathematical infinity: as an ideal (or variable) infinite, and as actual infinite—or completely determinate infinite [47]. I shall deal with these issues elsewhere. Yet the operational side was programmatically neglected in the emerging set theory.

The PM tacitly maintains double standards. Algebra was out of sync with geometry for centuries [3]. Yet surprising relationships exist between many disparate mathematical theories, notably between abstract linear forms and imaginary number systems [48]. Philosophizing mathematicians admitted complex numbers for use because they behaved like the real ones [49], as if the reals could set standard for more general complex tuples. The PM turned reasonings upside down, where particulars set standards for generals. It is as if corporals issuing orders to army generals. We do not lack ability to grasp abstract mathematical notions. The PM invented subtle ideas and discovered very sophisticated relationships. Yet its arrogant disregard for experimental evidence and suppression of objections made its achievements unverifiable. Absurd is maintained and paradoxes concealed in the PM, while nonsense is often passed on to physics. This is systemic problem that could and should be resolved. We should save pure mathematics from self-destruction and prevent further contamination of physics by arbitrarily postulated notions.

The most important issue at present is reconciliation of geometries with algebras, number theories, set theory and topologies. Formerly unexplained physical experiments and observations defy the mathematical infrastructure that the PM created for physics. This is not really new crisis, but the old one that was supposed to be swept "under the carpet" by formal axiomatization, so to say. The old crisis has resurfaced again, because PM rules by decree via existential postulates. I will deal here only with set theory just in order to prepare ground for addressing complex and hypercomplex numbers, and especially the very abstract operational role that is played by the imaginary unit. Although the imaginary unit is unintelligible insofar as its quantity is concerned, it was nevertheless admitted as a symbol for quite meaningful operation [50] and because operations on it are algebraically complete [51].

Even today PM and physics sometimes oversimplify imaginary numbers as being just references to second real dimension [52] rather than realization of the second dimension [53] that would underscore its operational as well as its functional status. Poncelet has already proposed that the imaginary unit identifies objects which may not be representable in the given reference frame [54], but his message has also been disregarded. The PM just cannot accept the possibility that there may be much more to physical reality than it could properly handle, and so it continues to stick with the age-old abstract formalism of complex numbers [55]. Complex numbers and the enhanced scope of abstract operations that they allow us to perform could be far more sophisticated than those performed on real numbers.

Fruitful exchange of letters between Einstein and Elie Cartan began when the latter complained about German scientists trying to "reinvent the wheel" [56]. This waste was due not only to some politically motivated animosities, but also to unverifiable, a priori standards implemented in the PM. Certain abstract mathematical ideas and consequently also some physical ones that were only *semi-developed* by some humble scientists, were probably far too advanced for the PM to digest them. By the Graham's economic law, which by the way has already been formulated by Copernicus, that says that bad money pushes more valuable coins off the market, those lofty ideas (often misunderstood even by their creators) were replaced by simplistic, usually arbitrarily postulated algebraic ideas. For this process to "succeed" the PM had to be kept at the obsolete level of debased ancient logic, however.

The mainstream PM still clings to the ancient, two-valued Aristotelian logic that is based upon three following main principles [57]:

- law of identity of things: $(A \equiv A)$;
- law of excluded middle: $(A \vee \neg A)$ [reads: either A or not A];
- law of contradiction: $\neg(A \wedge \neg A)$ [reads: not both A and not A];

where A is a thing, $\neg$ denotes unary logical negation 'not', $\wedge$ denotes binary logical operator 'and', and $\vee$ denotes binary logical operator 'or'. The two latter laws, and especially the law of excluded middle, though true as such, often cause inconsistencies when used at their face values. I will indicate departures from these laws. I have shown paradoxes that may be attributed to their misuse [2,3]. Although there are methodological [58] and axiomatic [59] prescriptions on how to keep logical consistency intact, there is no way to safely design absolutely consistent formal systems [60]. Therefore I have devised synthetic approach to mathematics and physics [2,3] just in order to prevent abstract, formal reasonings from spinning out of control. Synthetic mathematics requires constructions. For if one can build an abstract object, rather than just postulate its existence, then it can actually exist.

The problem of abstract mathematical existence is relevant to physics in respect to numbers too. Hamilton could not devise an abstract algebra of three-component numbers ("tertions") placed in between complex numbers and quaternions [61]. I have shown that operations on 3D vectors, which are algebraically represented as abstract 3-tuples of numbers, may actually span two dual 3D spaces, just in order to keep algebra consistent and in sync with geometry [3]. Our former presumption that abstract 3-tuples can always be identified with 3D vectors, and that the latter live inside a single 3D space is just unspoken postulate, oftentimes only dressed in an abstract axiomatic uniform. The PM is obsolete, but its abstract conceptual framework weighs down physics. We should overhaul geometries [3,62,63], algebras [2,3], theory of numbers and operations [2] and also the ingenious set theory (to be done in the present paper) in order to meaningfully pursue physics. At stake is huge backlog of unexplained experiments and observations.

2. Cantorian continuum hypotheses

Currently two major formulations of continuum hypotheses are pondered. The Cantorian continuum hypothesis (CCH) says that there are no sets of greater power than a denumerable set and of less power than the continuum, where the continuum referred to set of real numbers and denumerable sets were identified with either natural or integer numbers [1,64,65]. Generalized Continuum Hypothesis (GCH) is the conjecture that asserts that there are no sets whose cardinal numbers lie between the cardinal number $\aleph_0 = \infty$ of the set of natural numbers and the cardinal number $\aleph_1 = 2^{\aleph_0} = 2^{\infty}$ of the set of real numbers $\Re$ [1]. The GCH is not inconsistent with axioms of the ST [66] and therefore it can safely be added as an axiom to the ST, but it may seem somewhat irrelevant [67]. The ST is hardly constructible, because it does not give operational prescription on how to actually build sets (from their elements) that could be seamlessly operated on [67,68]. These two conjectures cannot be proved of disproved from the ST's axiomatic base [67]. But they could be disproved on their merit or perhaps because of lack thereof. The two hypotheses are essentially operational conjectures and thus they must adhere to relevant operational rules and procedures, by which we should decide their status. Georg Cantor treated his continuum hypothesis in terms of sets, whose unconstructiveness breaks their link to operations. The GCH is similarly postulative and thus also unconstructive. The GCH may be formulated in a different way [69]. The first two Cantor alephs appear much like the natural numbers zero and one [70].

I will show in what follows that the CCH and GCH are false. By denying the hypotheses one could open up the door to future non-Cantorian abstract set theories [71]. There are several mathematical, logical and even physical reasons for that. I have showed that one can operate on infinities just as on all the other numbers [2]. This act was prompted by the need for having operationally quite unrestricted abstract number system (NS) with somehow implemented division by zero. The reason for the implementation is that a mutispatial hyperspace has been invented [3,63] upon the abstract principle of duality, which requires an inverse transpose for dual linear vector spaces [63], which in turn could cause serious problems if the infamous prohibition of division by zero would still be in force. Since multispatiality of physical reality is *virtually* supported by special theory of relativity [62,63], and it has produced physically meaningful predictions and retrodictions that have already been confirmed in several formerly unexplained experiments [72] as well as strongly suggested in several others [73–76], I was quite sure that the nature was never really afraid of division by zero. To fully understand these issues, however, one would need new, synthetic approach to such structures. Mathematical topics of great importance to physics were handled by former PM in too abstract terms, hardly relevant to physical applications [13,77], or treated in terms of postulative morphisms [78,79]. If we want to avoid paradoxes we should use operational and synthetic methods. Mistakes and errors may always happen, but the PM needs balance checks that could intercept logical errors or identify mistakes that may breed paradoxes.

The multispatial structure of physical reality was virtually implied by few results of Elie Cartan [80], although he did not realize that [3,62], as well as by several mathematical and physical ideas of El Naschie. In particular, his approach to conjugate complex time [81–86] and to infinitely-dimensional fractal Cantorian spacetime [87–103] led to recent development of spatial structure of time flow [62,63] and consequently to discovery of nonradial potentials [72,73]. The nonradial effects of gravity—whose presence has been confirmed in experiments and observations conducted in space and on earth [72,74]—can generate an intrinsic gravitational repulsion [73], which could explain the observed, accelerated expansion of the whole universe. El Naschie superbly deals with physically meaningful fractal spacetime that is endowed with infinite number of dimensions. Georg Cantor has already recognized that in a topological sense dimensionality can be arbitrary large [104]. I shall show elsewhere that infinity is not something to be afraid of.

I should estimate the powers of various sets if I want to discuss them at a more exact, abstract mathematical rather than philosophical, level. In order to comprehend them I must use infinities to count actual powers of sets to be considered, even if this would be the last time I would ever touch infinities in my lifetime. I do not have to understand everything about them in order to use the infinities as operands. The only thing that I must take care of is that the abstract operations are performed unambiguously, according to group-theoretical rules devised for them. I do not need an exact knowledge about the sets under investigation, but I must identify their operational structures. As a matter of fact, I intend to discover most of their properties. One may be surprised to see that some truths of the Cantorian ST just cannot withstand the scrutiny of more exact, operational estimations.

There is nothing that I want to postulate right up front. Just to estimate the powers of sets and then I could draw some conclusions from these results that are for everyone to see and to repeat them. Operations on infinities are not about counting infinities, but about estimations. I should not, however, reduce squared or cubed infinity to just infinity, if such a rounding could have introduced a qualitative difference in their estimations. Instantaneous velocity and also infinitesimals were once ridiculed as "ghosts of departed quantities" [105]. Yet today, their formal representation is reinterpreted and they play fundamental roles in calculus. Taunting infinities may be fun, but we really need them for proper *modeling* of operational structures.

Truth in PM is a question of validity and consistency [106,107]. One may even say that truth is irrelevant to PM, because it is not about anything real [108]. However, if the abstract notions of ST can be made operational, then we should calculate or at least estimate all the values of its expressions and only after that we can talk about the truth of certain propositions that refer to these expressions. The main problem of the ST is that it was not operational and therefore the precise logical values of its propositions were difficult to evaluate. The other problem with the ST is that it came up with inadmissible generalizations that are common to paradoxes discovered in mathematics [109]. Mathematical abstractions should be effectively controlled.

3. Operational aspects of cardinal numbers

To fully understand the meaning of the CCH and GCH let us clean up the set-theoretical pig stall first. In order to compare cardinals of different sets I must estimate potential counts of members of these sets. Although infinity is not a regular number, I may use it as an abstract operand in estimations of potential counts. Let $\aleph_{00}$ denote the cardinal number of the set of all natural numbers. Since it corresponds to the natural denumerable infinity ∞ we get

$$\aleph_{00} = \infty \tag{1a}$$

as before. I am introducing new notation. We have the following count of representations of fractions from natural numbers within the interval (0,1):

$$\aleph_{01} = \infty \cdot (\infty - 1)/2 \tag{1b}$$

which is qualitatively of exactly the same class as $\aleph_{00}$. Yet because we have (natural) infinity of such intervals plus infinite count of naturals, then one can assign the whole set of rational numbers the following cardinal number:

$$\aleph_{02} = \infty \cdot (1 + \infty \cdot (\infty - 1)/2) = \infty^3/2 - \infty^2/2 + \infty \tag{2}$$

where fractions such as 1/2, 2/4 and 3/6 are distinct representations of the same fraction. Elimination of such redundant representations of the same value is possible, but it would not make any qualitative difference for the purpose of the present paper. It would reduce the estimate by about $\infty^3/4$, while still staying within the ∞^3 (cubed) general range. All redundant representations of the number one have been eliminated, however. Even though sums and products of infinities do result in compound infinities, from the operational standpoint the resultant infinities—when seen as some potential counts—are not quite identical to the ones operated on [2]. The common habit of rounding symbolically infinities to the very same infinity such as $\infty \cdot \infty = \infty^2 \approx \infty$ is quite appropriate in practice, but it could also derail one's ability to draw some subsequent conclusions. The first subscript denotes qualitative class of the aleph, whereas the second subscript denotes the aleph's quantitative class in this, slightly modified, new notation.

Although $\aleph_{02} > \aleph_{01} > \aleph_{00}$, $\aleph_{02}$ and $\aleph_{01}$ are denumerable—hence of the same qualitative class of infinity as $\aleph_{00}$. Potential counts of rationals are higher than the potential count of natural numbers alone, but the actual process of denumeration of their sets is only slightly different. If one would call the $\aleph_{00}$ linear sequential kind of infinity, then the $\aleph_{01}$ can be called half-matrix kind of infinity, and so $\aleph_{02}$ would appear as a 3D half-matrix, a sort of. The set of all rational numbers obviously has infinite, denumerable set of partitions, each of which has also denumerable infinite span. Hence the potential count of all its elements is equal to certain function of the natural infinity, which has the same type of cardinal number $\aleph_{02}$ as that of the natural numbers $\aleph_{00}$, even though these two cardinals are not identical. These two sets are thus similar, because the set of rationals with $\aleph_{02}$ elements is just as denumerable as the set of natural numbers is with only $\aleph_{00}$ elements. Cardinal numbers do refer to the potential count of elements in each particular set. Both the natural and rational numbers can be counted linearly and they both are sparse, but their respective sets' potential counts differ only quantitatively. Naturals are counted along straight line, whereas rationals are counted along a waving kind of line, which makes their estimated potential count a little bit higher. The cardinal numbers $\aleph_{01}$ and $\aleph_{02}$ are just estimates. I want to see their operational structure and their approximate ranges.

The interval $(0, 1)$, however, contains inverses of all rational numbers >1, some of which may be incommensurable (surds [110]), as well as infinite multitude of pure irrational numbers, which are potentially infinite decimals represented by unordered sequences of digits with possible repetitions. They cannot be reduced to any rational number. Nevertheless, the infinity of such purely irrational numbers within the interval $(0, 1)$ is of a different kind—because it is neither linear nor half-matrix-like sort of infinity anymore. The potential count of irrational numbers can be depicted thus by permutations with repetitions allowed. Since digital representation of any number can be always reduced to certain binary one, then the cardinal number of such a partition $(0,1)$ of the irrational numbers *alone* is determined by:

$$\aleph_{10} = 2^{\aleph_{00}} = 2^{\infty} = \infty. \tag{3}$$

which equals to all linearly infinite permutations with repetitions allowed [111]. The dotted infinity $\infty.$ denotes both irrational and real infinity [2]. Note that Georg Cantor assigned $\aleph_{10}$ to the whole set of real numbers. This was a big mistake, which happened because he did not operate on infinities. Had he estimated $\aleph_{01}$, he could have realized that it was mistake, although it is the Russell paradox that makes mistakes of this kind clearly visible.

Set with the cardinal number $\aleph_{10}$ is referred to as continuum, which is a dense multitude of numbers or points on a line. The irrational infinity is thus qualitatively different. Its potential count is the same as count of all subsets in an infinite discrete set. It sits at slightly higher level of abstraction in the hierarchy of cardinals, which are abstracts themselves. Former ST used to postulatively attribute the cardinal number $\aleph_{10}$ to the whole set of all real numbers and then proved that reals are as dense as the whole partition $(0,1)$. These two sets are of equal density, but of different configuration. In terms of enumerative mappings, the set of reals and its partition $(0,1)$ are uniquely mapped by inversion. However, the partition $(0,1)$ is double bound, while the set of all real numbers is half open (unbound). As with the stereographic projection—the closer to zero the denser the partition becomes. The unique mapping does not make the real partition $(0,1)$ quite identical with all reals.

When one constructs the set of reals operationally, from partitions such as $(0,1)$, however, the continuum of all reals may look slightly different. In some cases what one can see depends on how one approaches the issue. For Georg Cantor the difference between partition $(0,1)$ and the continuum of all reals was apparently negligible. In his time physical world was perceived as clockwork with no fuzziness, which is common place today. The primary reason for using infinity in our calculations is that we can compare various expressions and be able to see structural differences between them. When spoken of at the philosophical level, such differences are not visible.

Irrationals are distinct kind of numbers and quite different from rationals. Hence the cardinal number of the partition $(0,1)$, with rationals (and naturals included among them) and pure irrationals *together* is thus determined by

$$\aleph_{11} = \aleph_{01} + \aleph_{10} = \infty \cdot (\infty - 1)/2) + 2^{\infty} \tag{4}$$

which is a superposition of the rational and irrational cardinals. Yet the fact that we deal with superposition does not make the cardinality of the whole partition $(0,1)$ qualitatively different than the cardinality of irrationals within the partition. Superposition of sparse rationals does not make the density of irrationals within the partition $(0,1)$ qualitatively any denser. The continuum is just sparsely peppered with rationals, as if islands scattered over an ocean. Superposition of two distinct qualitative levels remains thus at the higher of these two qualitative levels.

Since we have infinitely many partitions like $(0,1)$, the cardinal number of the whole set of real numbers, or combined partitions—as opposed to just one single partition—is different from that of all the previous cardinals, for we got now linearly infinite multitude of all these partitions. Hence one can assign the set of all reals quite different, composite "real" cardinal number

$$\aleph_{20} = \aleph_{00} \cdot \aleph_{11} = \aleph_{00} \cdot (\aleph_{01} + \aleph_{10}) = \infty \cdot (\infty \cdot (1 + \infty \cdot (\infty - 1)/2) + 2^{\infty}) \tag{5}$$

that opens up new level of abstraction. Although the product $\aleph_{00} \cdot \aleph_{01}$ is does not make any qualitative change, the product $\aleph_{00} \cdot \aleph_{10}$ does raise the level of abstraction. It is as if transition from solar system—with continuum of dust and sparse, discrete ("infinite") multitude of planets, meteors, comets and all the other debris—to the higher level of whole universe with "infinitely" many, similar planetary systems. The cardinal number $\aleph_{20}$ is qualitatively and quantitatively different from all the lower cardinals. Georg Cantor did miss this tiny difference because he did not bother to estimate the cardinals. Superimposition of two distinct qualitative levels does generate qualitatively higher level. This is theoretically significant distinction.

From operational point of view, the irrational infinity $\infty.$ is the same as the real infinity, when it comes to representing any *single* irrational number. Note that each single irrational number has exactly the same representation as any real–irrational number, but the set of reals and the partition $(0,1)$ are not identical. From the set-theoretical point of view, the potential counts of sets of pure irrational numbers within the partition $(0,1)$ and of the whole set of all real numbers are quite different. Former ST missed this point entirely, because it did not construct sets, but merely postulated them. The simplest lesson from this investigation is that valid conclusions drawn about numbers must not be carried over onto their sets or subsets and probably vice versa.

Since former ST did not examine the difference between sets understood as mappings and constructive sets viewed as outcome of some operations on the sets' elements, it considered $\aleph_{10}$ as the cardinal number of the whole set of real numbers and of the real partition $(0,1)$ too. From strictly operational standpoint, however, sets are constructs.

Although former ST always talked about sets of some numbers, it virtually dealt with sets of some morphisms, because it assumed that sets are fully determined by some mappings, often as simple as a rule of selection in a procedural mapping. The ST declared sets and then arbitrarily postulated their properties, including their cardinal numbers, and after that it proved by derivations form some other arbitrary postulated axioms that its theorems were correct. However, the ST actually proved that if the ST's postulated axioms happened (by accident or a lucky guess, if you will) to be valid, then the proof was admissible. In majority of all other cases the ST and PM only proved that they were consistent, but not always that they were valid. Hence from postulated properties that happened to be false the PM dutifully deduced some false relationships, which it then hailed as truths, without even being aware of their falsity, of course.

The mix up of cardinals is thus the former ST's "cardinal" fault, because it refused to admit constructive sets. Since sets of numbers are supposed to be operational, one should construct them operationally. Although this is not the only way to construct them, it is preferable method from the SM's point of view. I have not created procedures for building sets in this note, but only outlined certain operational aspects of their construction for the purpose of our discussion. Nevertheless, one can see the difference that constructions can make. Operational constructions revealed quite unanticipated properties of the abstract sets under investigation and allowed us to discover their true nature. Synthetic operational approach reveals formerly invisible features.

The continuum of the all real numbers looks like set of sets, i.e., superset of partitions such as $(0,1)$. Also from this point of view it is a different kind of abstract continuum. The top-down approach of former ST gives different results than our bottom-up operational approach. To postulate that cardinal number of the whole set of all reals and that of the partition $(0,1)$ were quite identical could bring back the ghostly paradoxes that haunted mathematics at the beginning of 20th century. Former ST was badly mistaken in its tacit yet totally unchecked assumption of qualitatively equal densities of these two structurally different sets, which it has forgotten to estimate or measure. The unwarranted assumption was usually disguised as a proof of mapping-based equivalence of potential counts of these two sets, of course. Unique mappings may indicate similarity, but quality is not the same as similarity. Bohr's atom was similar to our solar system, yet of quite different quality.

By the same token the cardinal numbers of the sets of complex numbers, quaternions and octonions, respectively, could be denoted by

$$\aleph_{22} = (\aleph_{20})^2, \quad \aleph_{24} = (\aleph_{20})^4, \quad \aleph_{28} = (\aleph_{20})^8, \tag{6}$$

which are of the same qualitative type as that of all the real numbers. Georg Cantor was right when it comes to some qualitative aspects of the cardinals. Nevertheless, both the CCH and GCH are false. In the light of the relations (1–5) the cardinal numbers of the irrational partition $(0,1)$ and that of the whole set of real numbers are both qualitatively and quantitatively different, contrary to the CCH and GCH. Being independent of the other axioms of the ST, the falsity of these hypotheses does not really ruin the ST, which is not concerned with theoretical status of particular sets. Despite its obvious deficiencies, Georg Cantor has founded really formidable set theory, maybe slightly ahead of its time. Modern physics needs more reliable ST, however.

4. Foundation of non-Cantorian set theory

Ancient Greeks have discovered the irrational numbers while performing calculations for practical measurements of incommensurable line segments. According to a legend, Platonists allegedly even killed someone, who has revealed their undesirable discovery to the world outside. Ancient PM was declared perfect a priori by a postulative decree and so incommensurability might have been perceived as an imperfection on the idealistic image of the ancient pure mathematics. Today's PM has inherited the imperial trait from Platonists. It did not occur to them that the incommensurability could be a sure sign of much higher perfection than they could realize back then.

I can imagine that a higher class of hyperinfinity may also exist, namely

$$\aleph_{30} = (\aleph_{10})^{\aleph_{00}} = (\infty\cdot)^{\infty} \tag{7}$$

which may be called El Naschie cardinal, because it seemingly pertains to the infinitely-dimensional manifold of fractal Cantorian spacetime $E^{(\infty)}$, whose properties El Naschie relentlessly investigated. For its multilayered abstract topological structure seems to transcend more regular topological manifolds. Given its hyperspatial character and infinitely-dimensional nature one can span over that manifold a spacetime as well as other kind of vector spaces. It fits the bill for manifold to support the abstract multispatial hyperspace [2]. The El Naschie cardinal sits thus higher in the hierarchy of manifolds than the regular topological manifolds and so it fits structurally the fractal manifold too, which contains whole spaces among its elements [87]. Hence the infinitely-dimensional Cantorian spacetime is thus a sort of hyper-continuum—or hierarchical continuum of embedded continua.

There are infinitely many cardinals beyond the already mentioned ones [112]. They can be created inductively [113] the following way:

$$\aleph_{(x+1)p} = 2^{\aleph_{xp}} \tag{8}$$

at every qualitatively different level x with any quantitative level p. Their definition always makes greater cardinal than the previous one [114], of course. But the open question is whether or not a qualitatively different cardinal number can exist between such two cardinals obtained from the formula (8). Unlike Georg Cantor, however, I would hope to answer this question in affirmative. Since this is just a guess at this time, I would call this conjecture generalized hyper-continuum hypothesis (GHCH), and then consequently the simpler, particular hyper-continuum hypothesis (PHCH) may ask the simpler question: are there any qualitatively different cardinals between $\aleph_{2x}$ and $\aleph_{3x}$? By giving tentatively affirmative answers to both the GHCH and PHCH I am clearly betting on the sure winner ''horse'' called 'Progress'. Abstract mathematical infrastructure of our physical reality is so astonishingly diverse [3] that to bet otherwise would be arrogant.

The other—more specific—reason for my optimism is that some points could exist within the Cantorian spacetime manifold that somehow are not directly achievable, by analogy to sets, which are not Lebesgue measurable [115]. I shall discuss this issue elsewhere. Continuity is a functional feature [116] that is often defined in terms of mappings [117], whereas density of abstract spaces is usually defined in numbers' set-based terms of ST [118] or in terms of abstract points in topology [119]. Although topology is often called rubber geometry [120], its concepts are not that elastic. They do not cover all aspects of geometric spaces over topological manifolds [3,63].

In order not to needlessly obscure my reasonings, I have omitted the fact that numbers come in pairs—each pair with positive and negative copy of the given number, with the exception of the usual, natural zero (null). One could simply multiply all the estimates of alephs by two, if accounting for negative numbers were desired. I would rather view them as two different representations of basically the same number—hardly relevant to the ST.

5. Remark on paradoxes of the infinite

I will deal with issues of infinity elsewhere, but for the sake of the reader, who may be wondering how the division by zero and the new continuum hypotheses are affected by the alleged paradoxes of the infinite, let me show her/him how badly PM handled these issues in the past. A pretty good and representative example of such an alleged paradox of the infinite is given in [121] where the infinity is blamed for an absurd statement, namely that $1 = 2$

$$0 \cdot 1 = 0 \cdot 2 \Rightarrow 1(? = ?)2 \tag{8a}$$

which would be quite inadmissible if the implication would be correct, but it is not. The primary reason for writing the left-hand side (LHS) equation in this logically incomplete implication is an implicit (here) assertion that

$$0 \cdot 0 = 0 \tag{8b}$$

which was assumed as so self-evident that nobody contested it for the last 5000 years. Without assuming the Eq. (8b) one could not write the LHS in Eq. (8a). The correct formula for multiplication of zeros is given in Eq. (0b), of course. The formula (8b) defies the law of identity, but it still reigns supreme. In a sense all the aforementioned logical laws are tacitly set-aside in Eq. (8b). Infinity is not really the culprit here, but the perfect scapegoat—an eternal excuse that never fails us. Almost every mathematical fault could be successfully and shamelessly blamed on the infinity. Yet natural infinity could actually save the incomplete implication (8a) from the absurd, for if one would admit that $0 = 1/\infty$ then the whole nonsense simply disappears

$$0 = 1/\infty \Rightarrow 0 \cdot 1 = 1/\infty \neq 0 \cdot 2 = 2/\infty \tag{8c}$$

because the absurd implied in (8a) was actually caused by the unfortunate assumption (8b). This is not really a paradox of infinity, but a paradox of the arbitrarily postulated multiplication by zeros that went quite unchecked and unproved. The issue of the alleged paradoxes of the infinite was mistreated.

It was not my intention to provide negative example, but there is no other way to deal with such pure-mathematical deceptions. One does not have to be mathematically gifted to see the evasiveness of the logically incomplete implication (8a). Therefore we should deploy the synthetic approach to both mathematics and physics in order not to be persuaded by such miscarriages. I am not saying that PM purposely obstructed truth. Nonetheless, it certainly tried hard to suppress proposed corrections to its logical flops. For its own sake, mathematics should be open to constructive criticism and corrective amends, if necessary. The bill for the PM's faults is being paid by physics.

6. Summary and conclusions

In the light of the present investigation the CCH and GCH can appear as somewhat incorrectly posted conjectures, because their formulations were based upon imprecise estimates. They are vivid examples of the havoc that arbitrary postulates can cause in otherwise perfectly valid reasonings. The two hypotheses as well as the former ST were founded upon notions whose connotations were overly influenced by philosophy. Former ST introduced several abstract notions that were formally valid, but not really constructed from operations or otherwise. They have been either explicitly postulated in arbitrary axioms, or perhaps implicitly assumed by some unwarranted generalizations or poorly designed abstractions. By using natural infinity as an abstract operand I have estimated the potential counts of sets and from them drawn conclusions concerning their qualitative aspects.

If taken literally in its traditional context, the CCH is false and so is also the GCH. Although I can see how difficult it may be to come up with quite adequate definitions of some subtle abstract concepts, yet I would prefer to avoid arbitrary existential postulates in lieu of constructions. The idea that infinity could be defined by a feature of the sets whose proper subsets are of equal power, was already floated by Galileo and presumably even earlier. Such a definition could have been made constructive, for enumeration is a procedural concept. From the very moment when Dedekind proposed it as definition of infinity in the ST and then put as an example his allegedly infinite train of thoughts, however, the ST virtually ceased to investigate sets given in actual mathematical reality and started chasing after plethora of ghosts-like abstract notions, vaguely implied by that fancy definition.

Falsity of the continuum hypotheses does not depreciate the overall value of the ST. Actually it seems very advantageous to have them proven false, because the allegedly stable picture of abstract mathematical reality gains quite unanticipated diversity. Unlike the Cantorian hypotheses, these new ones conjecture existence of certain distinct, mixed hyper-continuum that may serve as the manifold over which dual pairs of linear vector spaces with quite unrestricted abstract operations could be effectively constructed. Such unrestricted operability is desirable for the abstract multispatial hyperspace whose existence is suggested by many formerly unexplained experiments.

It is often believed that Georg Cantor went mad because the CCH he was working on was theoretically quite hopeless, undecidable problem [122]. Although it could not be decided on the axiomatic basis of the Cantorian set theory, its falsity is pretty clear from the synthetic operational standpoint. Axioms are usually so postulated as to make proofs of most, rather simple, properties easier. To derive something entirely new from such postulated axioms is hardly possible. Therefore constructive synthetic approach is necessary to both: mathematics and physics. Experimentally verifiable syntheses may transcend superficial axiomatic restrictions.

References

[1] Suppes P. Axiomatic set theory. New York: Dover Publications; 1972. p. 109, 193.
[2] Czajko J. On Cantorian spacetime over number systems with division by zero Chaos, Solitons & Fractals [to appear].
[3] Czajko J. Elie Cartan and pan-geometry of multispatial hyperspace. Chaos, Solitons & Fractals 2004;19:479–502.
[4] Stoll RR. Set theory and logic. New York: Dover Publications; 1979. p. 80.
[5] Gemignani MC. Elementary topology. Reading, MA: Addison-Wesley; 1972. p. 116.
[6] Lin S-YT, Lin Y-F. Set theory: an intuitive approach. Boston: Houghton-Mifflin; 1974. p. 85.
[7] Kurosh AG. Lectures on general algebra. New York: Chelsea Publications; 1965. p. 30.
[8] Kurosh AG. The theory of groups II. New York: Chelsea Publications; 1956. p. 233.
[9] van der Waerden BL. Algebra I. New York: Frederick Ungar Publications; 1970. p. 32.
[10] McCoy NH. Rings and ideals. The Math Association of America; 1962. p. 1.
[11] Miller KS. Elements of modern abstract algebra. New York: Harper and Brothers Publishers; 1958. p. 54.
[12] Katz R. Axiomatic analysis. Boston: D.C. Heath and Co; 1964. p. 121.
[13] Cartan E. Les groupes bilinéaires et les systèmes de nombres complexes. In: Cartan E, editor. Oeuvres complètes T 1 p 2. Paris: Gauthier-Villars; 1953. p. 7–105.
[14] Cartan E. Sur les groupes linéaires quaternioniensis. In: Cartan E, editor. Oeuvres complètes. Paris: Gauthier-Villars; 1953. p. 251–63.
[15] Banach S. Théorie des opérations linéaires. In: Banach S, editor. Oeuvres II. Warszawa: PWN; 1979. p. 13–219 [p. 42].
[16] Banach S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. In: Banach S, editor. Oeuvres II. Warszawa: PWN; 1979. p. 305–48 [p. 306].
[17] Banach S. Die Theorie der Operationen und ihre Bedeutung für die Analysis. In: Banach S, editor. Oeuvres II. Warszawa: PWN; 1979. p. 434–41 [p. 436].

[18] Hazewinkel M. Formal groups and applications. New York: Academic Press; 1978. p. 147.
[19] Hall Jr M. The theory of groups. New York: Macmillan; 1959. p. 29.
[20] Zassenhaus HJ. The theory of groups. New York: Chelsea Publications; 1958. pp. 35, 45.
[21] Alexandroff PS. An introduction to the theory of groups. New York: Hafner Publishing; 1959. p. 77.
[22] McCarthy G. Topology. In: An introduction wih applications to topological groups. New York: Dover Publications; 1988. p. 87.
[23] Lorenzen P. Ein vereinfaches Axiomsystem für Gruppen. J reine angew Math 1940;182:50.
[24] Swokowski EW. Calculus. Boston: PWS Publications; 1992. p. 40.
[25] Seife C. Zero. In: The biography of a dangerous idea. New York: Penguin; 2000. p. 2.
[26] Pedoe D. Geometry. In: A comprehensive course. New York: Dover Publications; 1998. p. 206.
[27] Brannan DA, Esplen MF, Gray JJ. Geometry. Cambridge: Cambridge University Press; 1998. p. 218.
[28] Yaglom IM. Geometric transformations III. New York: Random House; 1973. p. 55.
[29] Halmos PR. Naïve set theory. Princeton, NJ: D. Van Nostrand Co; 1960. p. 53.
[30] Holmes MR. Strong axioms of infinity in NFU. J Symb Logic 2001;66:87–116.
[31] Beth EW. Semantic entailment and formal derivability. In: Hintikka J, editor. The philosophy of mathematics. Oxford: Oxford University Press; 1969. p. 9–41 [see p. 39].
[32] Rucker R. Infinity and the mind. Princeton, NJ: Princeton University Press; 1995. pp. 5, 43.
[33] Dedekind R. Essays on the theory of numbers. New York: Dover Publications; 1963. p. 11.
[34] Fehr J. Introduction to the theory of sets. Englewood Cliffs, NJ: Prentice-Hall; 1958. p. 18.
[35] Zehna PW. Sets with applications. Boston: Allyn and Bacon; 1966. p. 273.
[36] Hamilton NT, Landin J. Set theory. In: The structure of arithmetic. Boston: Allyn and Bacon; 1961. p. 101.
[37] Lang S. Algebraic structures. Reading, MA: Addison-Wesley; 1968. p. 169.
[38] Lakoff G, Núñez RE. Where mathematics comes from. New York: Basic Books; 2000. p. 180.
[39] Cantor G. Ueber eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen. J Reine Angew Math 1874;77:258–62.
[40] Cantor G. Sur divers théorèmes de la théorie des ensembles de points situes dans un espace continu à N dimensions. Acta Math 1883;1:409–14.
[41] Cantor G. Ueber unendliche lineare Punktmannigfaltigkeiten III. Math Ann 1882;20:113–21.
[42] Cantor G. Sur les ensembles infinis et linéaires de points. I–IV. Acta Math 1883;2:349–80.
[43] Cantor G. Ueber unendliche lineare Punktmannigfaltigkeiten IV. Math Ann 1883;21:51–8.
[44] Cantor G. Über verschiedene Theoreme aus der Theorie der Punctmengen in einem n-fach ausgedehnten stetigen Raume G_n. Zweite Mitteilung Acta Math 1885:105–24.
[45] Cantor G. De la puissance des ensembles parfaits du points. Acta Math 1884;4:381–92.
[46] Cantor G. Ueber unendliche lineare Punktmannigfaltigkeiten VI. Math Ann 1884;23:453–88.
[47] Cantor G. Transfinite numbers. [Bingley GA. In: The Classics of the St. John's Program, Trans.]; 1941. p.94.
[48] Hecke E. Mathematische Werke. Göttingen: Vandenhoeck and Ruprecht; 1983. p. 418.
[49] Frege G. In: McGuiness B, editor. Collected papers on mathematics, logic and philosophy. New York: Basil Blackwell; 1984. p. 2.
[50] Clifford WK. The common sense of the exact sciences. New York: Dover Publications; 1955. p. 172.
[51] Clifford WK. On the classification of geometric algebras. In: Clifford Tucker R, editor. Clifford mathematical papers. New York: Chelsea Publications; 1968. p. 397–401.
[52] Conant DR. Unified quantum field theory. Los Alamos, NM; 1978. p. 136.
[53] Lloyd AH. The meaning of $\sqrt{-1}$. J Philos Psych Sci Methods 1908;5:141–50.
[54] Poncelet JV. Traité des propriétés projectives des figures. Paris; 1822. p. 28.
[55] Weil A. On the theory of complex multiplication. In: Weil A, editor. Oeuvres Scientifiques II. New York: Springer-Verlag; 1979. p. 263–76.
[56] Debever R, editor. Elie Cartan and Albert Einstein: letters on absolute parallelism. Princeton, NJ: Princeton University Press; 1979.
[57] Stewart BM. Theory of Numbers. New York: Macmillan; 1964. p. 212.
[58] Tarski A. Introduction to logic and the methodology of deductive sciences. New York: Dover Publications; 1995. p. 135.
[59] Gentzen G. Die gegenwärtige lage in der mathematischen Grundlagenforschung. Neue Fassung des Widerspruchsfreiheitsbeweis für die reine Zahlentheorie. Hildesheim; 1970. p. 13.
[60] Gödel K. On formally undecidable propositions of principia mathematica and related systems I. New York: Dover Publications; 1992. p. 1, 40.
[61] Kline M. Mathematical thought from ancient to modern times II. New York: Oxford Univ Press; 1990. p. 777.
[62] Czajko J. On conjugate complex time I: complex time implies existence of tangential potential that can cause some equipotential effects of gravity. Chaos, Solitions & Fractals 2000;11:1983–92.
[63] Czajko J. On conjugate complex time III: superstrings and complex Lorentz transformation. Chaos, Solitons & Fractals 2001;12:951–67.
[64] Cantor G. Beiträge zur begründung der transfiniten mengenlehre I. Math Ann 1895;46:481–512.
[65] Cantor G. Beiträge zur begründung der transfiniten mengenlehre II. Math Ann 1897;49:207–46.
[66] Gödel K. The consistency of the continuum hypothesis. Princeton, NJ: Princeton Univ. Press; 1940. p. 61.
[67] Cohen PJ. Set theory and the continuum hypothesis. New York: Freeman; 1966. p. 2, 85.

[68] Gray JJ. The Hilbert challenge. Oxford: Oxford University Press; 2000. p. 223.
[69] Sierpinski W. Sur une propriétè de la droite équivalente á l'hypothèse du continu. Ganita 1954;5:113–6.
[70] Dunham W. Journey through genius. In: The Great Theorems of Mathematics. New York: Wiley; 1990. p. 282.
[71] Stewart I. Concepts of modern mathematics. New York: Dover Publications; 1995. p. 297.
[72] Czajko J. On conjugate complex time II: equipotential effect of gravity retrodicts differential and predicts apparent anomalous rotation of the sun. Chaos, Solitons & Fractals 2000;11:2001–16.
[73] Czajko J. Intrinsic gravitational repulsion. Chaos, Solitons & Fractals 2004;20:683–700.
[74] Czajko J. Complex conjugate time in macro experiments. Chaos, Solitons & Fractals 2002;13:17–23.
[75] Czajko J. On the Hafele–Keating experiment. Ann Phys (Leipzig) 1990;47:517–8.
[76] Czajko J. Experiments with flying atomic clocks. Exp Technol Phys 1991;39:145–7.
[77] Hausmann J-C, Vogel P. Geometry on Poincaré spaces. Princeton, NJ: Princeton University Press; 1993. p. 14.
[78] Bourbaki N. Eléments de mathématique: groupes et algèbres de Lie. Paris: Hermann; 1960. p. 48.
[79] Cordaro PD. Hyperfunctions on hypo-analytic manifolds. Princeton, NJ: Princeton University Press; 1994. p. 29.
[80] Cartan E. La méthode du repère mobile, la théorie des groupes continues et les espaces généralisés. Paris: Hermann; 1935. p. 39.
[81] El Naschie MS. On the nature of complex time, diffusion and the two-slit experiment. Chaos, Solitons & Fractals 1995;5:1031–2.
[82] El Naschie MS. On conjugate complex time and information in relativistic quantum theory. Chaos, Solitons & Fractals 1995;5:1551–5.
[83] El Naschie MS. Time symmetry breaking, duality and Cantorian spacetime. Chaos, Solitons & Fractals 1996;7:499–518.
[84] El Naschie MS. Wick rotation, Cantorian spaces and the complex arrow of time in quantum physics. Chaos, Solitons & Fractals 1996;7:1501–6.
[85] El Naschie MS. A note on quantum gravity and Cantorian spacetime. Chaos, Solitons & Fractals 1997;8:131–3.
[86] El Naschie MS. Fractal gravity and symmetry breaking in a hierarchical Cantorian spacetime. Chaos, Solitons & Fractals 1997;8:1865–72.
[87] El Naschie MS. Hyperdimensional geometry and the nature of physical spacetime. Chaos, Solitons & Fractals 1999;10:155–8.
[88] El Naschie MS. Introduction to nonlinear dynamics, general relativity and the quantum. Chaos, Solitons & Fractals 1997;8:VII–X.
[89] El Naschie MS. On the irreducibility of spatial ambiguity in quantum physics. Chaos, Solitons & Fractals 1998;9:913–9.
[90] El Naschie MS. Renormalization semi-groups and the dimension of Cantorian spacetime. Chaos, Solitons & Fractals 1994;4:1141–5.
[91] El Naschie MS. Dimensional symmetry breaking and gravity in Cantorian space. Chaos, Solitons & Fractals 1997;8:753–9.
[92] El Naschie MS. Radioactive decay and the structure of $E^{(\infty)}$ quantum spacetime. Chaos, Solitons & Fractals 1999;10:17–23.
[93] El Naschie MS. Penrose universe and Cantorian spacetime as a model for noncommutative quantum geometry. Chaos, Solitons & Fractals 1998;9:931–3.
[94] El Naschie MS. The fractal dimension of spacetime—remarks on theoretical derivation and experimental verification. Chaos, Solitons & Fractals 1998;9:1211–7.
[95] El Naschie MS. Knot complement with a three sphere volume as a model for $E^{(\infty)}$ spacetime. Chaos, Solitons & Fractals 1998;9:1787–8.
[96] El Naschie MS. Von neumann geometry and $E^{(oo)}$ quantum spacetime. Chaos, Solitons & Fractals 1998;9:2023–30.
[97] El Naschie MS. Superstrings, knots and noncommutative geometry in $E^{(\infty)}$ space. Int J Theor Phys 1998;37:2935–51.
[98] El Naschie MS. Jones' invariant Cantorian geometry and quantum spacetime. Chaos, Solitons & Fractals 1999;10:1241–50.
[99] El Naschie MS. Quantum groups and hamiltonian sets on a nuclear spacetime Cantorian manifold. Chaos, Solitons & Fractals 1999;10:1251–6.
[100] El Naschie MS. On the unification of heterotic strings, m theory and $E^{(\infty)}$ theory. Chaos, Solitons & Fractals 2000;11:2397–408.
[101] El Naschie MS. Kleinian Groups in $E^{(\infty)}$ and their connection to particle physics and cosmology. Chaos, Solitons & Fractals 2003;16:637–49.
[102] El Naschie MS. Modular groups in Cantorian $E^{(\infty)}$ high-energy physics. Chaos, Solitons & Fractals 2003;16:353–66.
[103] El Naschie MS. The Cantorian interpretation of high energy physics and the mass spectrum of elementary particles. Chaos, Solitons & Fractals 2003;17:989–1001.
[104] Cantor G. Une contribution a la théorie des ensembles. Acta Math 1883;2:311–28.
[105] Boyer CB. The history of calculus and its conceptual development. New York: Dover; 1959. p. 200, 227.
[106] Smart JR. Introductory geometry. An informal approach. Belmont: Brooks/Cole; 1967. p. 181.
[107] Janiszewski Z. Sur le réalisme et l'idéalisme en mathématiques. In: Janiszewski Z, editor. Oeuvres choisies. Warszawa: PWN; 1962. p. 309–17.
[108] Delong H. A profile of mathematical logic. Reading, MA: Addison-Wesley; 1970. p. 58.
[109] Meschkowski H. Hundert Jahre Mengenlehre. München: Deutsche Taschenbuch-Verlag; 1973. p. 52.
[110] Chrystal G. Algebra. In: An Elementary Textbook I. New York: Chelsea Publications; 1964. p. 203.
[111] MacAtee P, Rice T, Whidden J. Generalized permutations and combinations. Available from: <http://swid.unl.edu/csce235/section4_6_3.html(Oct-9-2003)>.
[112] Simmons GF. Introduction to topology and modern analysis. New York: McGraw-Hill; 1963. p. 42.
[113] Eccles PJ. An introduction to mathematical reasoning. Cambridge: Cambridge University Press; 2002. p. 49.
[114] Kelly JL. General topology. Princeton, NJ: D. Van Nostrand; 1964. p. 280.

[115] Halmos PR. Measure theory. Princeton, NJ: D. Van Nostrand; 1964. p. 69.
[116] Mendelson B. Introduction to topology. Boston: Allyn and Bacon; 1963. p. 103.
[117] Newman MHA. Elements of the topology of plane sets of points. Cambridge: The University Press; 1964. p. 66.
[118] Niven I. Irrational numbers. The Math Association of American Press; 1956. p. 5.
[119] Hocking JG, Young GS. Topology. New York: Dover Publications; 1988. p. 11.
[120] Chinn WG, Steenrod NE. First concepts of topology. The Math Association of American Press; 1966. p. 57.
[121] Northrop EP. Paradoxes in mathematics. In: A book of paradoxes. Princeton, NJ: D. Van Nostrand; 1944. p. 78.
[122] Aczel AD. The mystery of the aleph. New York: Washington Square Press; 2001. p. 155.