# Chapter 1

# A Characterization of Twin Prime Pairs

PROCEEDINGS—NCUR V. (1991), VOL. I, PP. 362–366.

**Jeffrey F. Gold**

*Department of Mathematics, Department of Physics*
*University of Utah*

**Don H. Tucker**

*Department of Mathematics*
*University of Utah*

The basic idea of these remarks is to give a tight characterization of twin primes greater than three. It is hoped that this might lead to a decision on the conjecture that infinitely many twin prime pairs exist; that is, number pairs $(p, p+2)$ in which both $p$ and $p+2$ are prime integers.

The basic idea of the arguments is to decompose the integers *modulo* 6 and then subtract from this set all composites, leaving only primes, then discard further leaving only twin primes pairs.

Central to all the arguments and the basic idea driving this process is the function

$$G(n) = 3n + 2^{sin^2(\frac{n\pi}{2})}$$

generated by the first author. It is easily seen that

$$G(2k) = 6k + 1 \in 1 \bmod 6$$

and

$$G(2k+1) = 6k + 5 \in 5 \bmod 6$$

and indeed $G$ exhausts $1$ *mod* $6 \cup 5$ *mod* $6$.

Let us now consider the classes of integers *modulo* 6 and make some elementary, though useful, observations.

| | | | | | |
|---|---|---|---|---|---|
| 0 mod 6 | $\cong$ | $6k$ | | | *even and composite* |
| 1 mod 6 | $\cong$ | $6k+1$ | | | *odd* |
| 2 mod 6 | $\cong$ | $6k+2$ | $=$ | $2(3k+1)$ | *even and composite* (*except* $k=0$) |
| 3 mod 6 | $\cong$ | $6k+3$ | $=$ | $3(2k+1)$ | *composite* (*except* $k=0$) |
| 4 mod 6 | $\cong$ | $6k+4$ | $=$ | $2(3k+2)$ | *even and composite* |
| 5 mod 6 | $\cong$ | $6k+5$ | | | *odd* |

Therefore, all primes, except 2 and 3, are in $1$ *mod* $6 \cup 5$ *mod* $6$ and thus are all generated by the function $G(n)$.

Next, we will find a way to delete exactly the composites from $1$ *mod* $6$ and $5$ *mod* $6$. To that end, we construct a multiplication table for the counting numbers in *modulo* 6 arithmetic. For simplicity, we denote $n$ *mod* $6$ as $n$ for $n \in \{0,1,2,3,4,5\}$.

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 |
| 2 | 0 | 2 | 4 | 0 | 2 | 4 |
| 3 | 0 | 3 | 0 | 3 | 0 | 3 |
| 4 | 0 | 4 | 2 | 0 | 4 | 2 |
| 5 | 0 | 5 | 4 | 3 | 2 | 1 |

A brief examination of the table shows that $1$ *mod* $6 \cup 5$ *mod* $6$ is closed under multiplication; moreover, the only composites possible in $1$ *mod* $6 \cup 5$ *mod* $6$ are made of factors already in $1$ *mod* $6 \cup 5$ *mod* $6$. Stated somewhat differently, each composite element in $1$ *mod* $6 \cup 5$ *mod* $6 \equiv \{G(n)\}_{n=1}^{\infty}$ has a (not necessarily unique) factorization of one of the following three forms:

$$\begin{array}{lcll} (5+6i)(5+6j) & \in & 1 \text{ mod } 6 & i,j \geq 0 \\ (7+6i)(5+6j) & \in & 5 \text{ mod } 6 & i,j \geq 0 \\ (7+6i)(7+6j) & \in & 1 \text{ mod } 6 & i,j \geq 0 \end{array}$$

**Remark** In the products above, we have chosen to consider $1$ *mod* $6$ as $7$ *mod* $6$ so as to allow both $i$ and $j$ to begin at 0.

Now, suppose that $G(n_1) \in 1$ *mod* $6$ is composite, so that one of two forms must hold; either

$$G(n_1) = 3n_1 + 1 = (5+6i)(5+6j) = 25 + 30i + 30j + 36ij$$

so that

$$n_1 = 8 + 10i + 10j + 12ij \ , \tag{1.1}$$

or else it takes the form (consulting the table)

$$G(n_3) = 3n_3 + 1 = (7+6i)(7+6j) = 49 + 42i + 42j + 36ij$$

in which case

$$n_3 = 16 + 14i + 14j + 12ij \ . \tag{1.2}$$

Finally, if $G(n_2) \in 5 \ mod \ 6$, then we must have

$$G(n_2) = 3n_2 + 2 = (7+6i)(5+6j) = 35 + 30i + 42j + 36ij$$

or

$$n_2 = 11 + 10i + 14j + 12ij \ . \tag{1.3}$$

Therefore, the set of all composites in $1 \ mod \ 6 \ \cup \ 5 \ mod \ 6$ is exactly the set

$$\{G(n_1)\} \ \cup \ \{G(n_2)\} \ \cup \ \{G(n_3)\}$$

where

$$\begin{cases} n_1 &= 8 + 10i + 10j + 12ij \quad i,j \geq 0 \\ n_2 &= 11 + 10i + 14j + 12ij \quad i,j \geq 0 \\ n_3 &= 16 + 14i + 14j + 12ij \quad i,j \geq 0 \end{cases}$$

**Theorem 1** *The prime integers > 3 are exactly*

$$\{G(n)\} \setminus \{\{G(n_1)\} \ \cup \ \{G(n_2)\} \ \cup \ \{G(n_3)\}\} \ .$$

Now that we have captured the primes and separated out the composites, we begin our search for a characterization of the twin primes.

**Observations:**

(i) Suppose $k$ is even, then $k+1$ is odd, and therefore (from above), $G(k) = 3k+1$ and $G(k+1) = 3k+5$, so that

$$G(k+1) - G(k) = 4 \ .$$

Therefore, the pair $[G(k), G(k+1)]$ does not form a twin prime pair if $k$ is an even integer.

(ii) Suppose $k$ is odd, then $k+1$ is even and hence $G(k) = 3k+2$ and $G(k+1) = 3k+4$, so that $G(k+1) - G(k) = 2$ and we have the possibility of a twin prime pair.

**Upshot** $G$ generates all the primes $> 3$, indeed all of $1 \; mod \; 6 \cup 5 \; mod \; 6$, and if a pair $[G(n), G(n+1)]$ is to be a twin prime pair, $n$ must be an odd integer.

The question before us is how to cast out those odd integers $n$ for which we do not obtain prime pairs. Recall that $n$ odd implies that $G(n) \in 5 \; mod \; 6$ so that a twin prime pair $(P, P+2)$ has the property that $P \in 5 \; mod \; 6$ while $P + 2 \in 1 \; mod \; 6$.

Now let us suppose that $n$ is odd and $G(n)$ is not in $\{G(n_1)\} \cup \{G(n_2)\} \cup \{G(n_3)\}$, i.e., we have a prime $G(n)$ which is a qualified candidate for the first of a twin prime pair.

The integers $\{n_1\}$ and $\{n_3\}$ given by (1.1) and (1.2) are all even, hence already excluded since n is odd, therefore we are really saying $n \notin \{n_2\}$, the set generated by (1.3), i.e.,

$$n \neq 11 + 10i + 14j + 12ij = n_2 \; .$$

Moreover, if we are to have a twin prime pair, $n + 1$ must be even and not in $\{n_1\}$ nor $\{n_3\}$, i.e.,

$$\begin{cases} n+1 & \neq & 8 + 10i + 10j + 12ij \\ n+1 & \neq & 16 + 14i + 14j + 12ij \end{cases}$$

$$\Rightarrow \begin{cases} n & \neq & 7 + 10i + 10j + 12ij & = & n_4 \\ n & \neq & 15 + 14i + 14j + 12ij & = & n_5 \end{cases}$$

Collectively, this says that if $n$ is odd and not in any of the three forms

$$(\dagger) \quad \begin{cases} 7 + 10i + 10j + 12ij = n_4 \\ 11 + 10i + 14j + 12ij = n_2 \\ 15 + 14i + 14j + 12ij = n_5 \end{cases}$$

then $n+1$ is even, neither $G(n)$ nor $G(n+1)$ is in $\{G(n_1)\} \cup \{G(n_2)\} \cup \{G(n_3)\}$ and therefore both $G(n)$ and $G(n+1)$ are prime and $G(n+1) - G(n) = 2$ and we are left with a twin prime pair.

**Theorem 2** *Any twin prime pair greater than* $(3, 5)$ *is of the form* $[G(n), G(n+1)]$ *where n is odd and not of the forms* $(\dagger)$ *and conversely, for any such odd integer* $n$*, the pair* $[G(n), G(n+1)]$ *is a twin prime pair.*

The central question then is whether there exist infinitely many odd numbers which cannot be written in one of the forms $(\dagger)$.