Appendix to Part Zero

This is Liberty-hall, gentlemen!
Oliver Goldsmith, "She Stoops to Conquer"

In this appendix we informally discuss the formalisation of our theory, with particular regard to the nature of the inductions involved.

In Chapter 3 we gave a formal definition of the *birthday* of an arbitrary number, and we suspect that many readers would have felt happier had we described all our inductive arguments in terms of birthdays. The typical induction would then read:

"If $P(y)$ holds for all y with birthdays less than the birthday of x, then $P(x)$ holds. So by induction, $P(x)$ holds for all x."

The feeling that this sort of treatment adds to the precision of an inductive argument is much too common, and is responsible for the introduction of many irrelevancies in the literature. Thus in the case under discussion the notion of birthday is completely irrelevant, and all that is needed to justify the induction is the principle:

"If P is some proposition that holds for x whenever it holds for all x^L and x^R, then P holds universally."

We have already remarked that this was what we intended to be understood from the last sentence of our construction: "All numbers are constructed in this way."

The general inductive principle above has for its counterpart in the Zermelo–Fraenkel set theory ZF the so-called axiom of restriction, or foundation, which can be stated in the form:

"If P is some proposition that holds of a set x whenever it holds for all members of x, then P holds for every set."

Perhaps part of the prejudice against inductive arguments with arbitrary inductive variable is that this axiom is usually only stated in the peculiarly opaque form:

"Any non-empty Class X has some member disjoint from X."

It is then proved equivalent to the assertion that every set belongs to some set P_α, where these are the sets obtained by transfinite iteration of the power-set construction (P_α being the union of the power-sets of all P_β ($\beta < \alpha$)), but not to the more general inductive principle stated above. To see the latter equivalence, we need only take X as the Class of all x for which P fails.

The mention of ZF prompts a discussion of the problems of formalising our theory within ZF. Some people sense difficulties associated with the fact that an equality class of numbers is naturally a proper Class, rather than a set, and so cannot serve as an element in some other class. A slightly greater knowledge of ZF shows that this is no obstacle, and the theory may be formalised along the following lines.

We define a *game* as an ordered pair $\langle L, R\rangle$ of sets whose elements are themselves games of lower rank. (The *rank* of a set is the least α for which that set belongs to P_α.) Then we introduce the relation $\leqslant$ on games by

$$x \leqslant x' \text{ iff (no member of } L \text{ is } \geqslant x', \text{ and } x \geqslant \text{ no member of } R'),$$

where $x = \langle L, R\rangle$, and $x' = \langle L', R'\rangle$. The equivalence relation $\doteq$ is then introduced by $x \doteq y$ if and only if $x \leqslant y$ and $y \leqslant x$, and *prenumbers* are then defined inductively by the requirement that every member of $L \cup R$ should be a prenumber, and no member of $L \geqslant$ any member of R.

The fact that the equivalence classes of $\doteq$ may be proper Classes is then overcome by the standard dodge—for any x we define $[x]$ to be the set of all y *of the least possible rank* that are equivalent to x. Any set of the form $[x]$ for some prenumber x is then called a *number*.

So a number becomes a rather curiously restricted set of ordered pairs $\langle L, R\rangle$, each of which is of course a set according to the Kuratowski definition $\langle L, R\rangle = \{\{L\}, \{L, R\}\}$.

Another, and technically simpler, approach makes use of the sign-expansions introduced in Chapter 3. We define a number to be its sign-expansion, which is of course a function from some ordinal α to the set $\{+, -\}$. We then define order-relations in terms of these expansions by the rules in Chapter 3, and *define* $\{L \mid R\}$ to be the simplest (i.e. *shortest*) number greater than every member of L and less than every member of R. We then define the arithmetic operations by the formulae in Chapter 0.

In this simpler formalisation, a number is still a pretty complicated thing, namely a certain function in ZF, which is of course a certain set of Kuratowskian ordered pairs. The first members of these ordered pairs will be ordinals in the sense of von Neumann, and the second members chosen from the particular two-element set we take to represent $\{+, -\}$.

The curiously complicated nature of these constructions tells us more about the nature of formalisations within ZF than about our system of numbers, and it is partly for this reason that we did not present any such formalised

theory in this book. But the main reason was that we regard it as almost self-evident that our theory is as consistent as ZF, and that formalisation in ZF destroys a lot of its symmetry. Plainly the proper set theory in which to perform a formalisation would be one with two kinds of membership, and would in fact be very like the abstract theory of games that underlies the next part of this book.

It seems to us, however, that mathematics has now reached the stage where formalisation within some particular axiomatic set theory is irrelevant, even for foundational studies. It should be possible to specify conditions on a mathematical theory which would suffice for embeddability within ZF (supplemented by additional axioms of infinity if necessary), but which do not otherwise restrict the possible constructions in that theory. Of course the conditions would apply to ZF itself, and to other possible theories that have been proposed as suitable foundations for mathematics (certain theories of categories, etc.), but would not restrict us to any particular theory. This appendix is in fact a cry for a Mathematicians' Liberation Movement!

Among the permissible kinds of construction we should have:

(i) Objects may be created from earlier objects in any reasonably constructive fashion.
(ii) Equality among the created objects can be any desired equivalence relation.

In particular, set theory would be such a theory, sets being constructed from earlier ones by processes corresponding to the usual axioms, and the equality relation being that of having the same members. But we could also, for instance, freely create a new object (x, y) and call it the ordered pair of x and y. We could also create an ordered pair $[x, y]$ different from (x, y) but co-existing with it, and neither of these need have any relation to the set $\{\{x\}, \{x, y\}\}$. If instead we wanted to make (x, y) into an unordered pair, we could define equality by means of the equivalence relation $(x, y) = (z, t)$ if and only if $x = z, y = t$ *or* $x = t, y = z$.

I hope it is clear that this proposal is not of any particular theory as an alternative to ZF (such as a theory of categories, or of the numbers or games considered in this book). What is proposed is instead that we give ourselves the freedom to create arbitrary mathematical theories of these kinds, but prove a metatheorem which ensures once and for all that any such theory could be formalised in terms of any of the standard foundational theories.

The situation is analogous to the theory of vector spaces. Once upon a time these were collections of n-tuples of numbers, and the interesting theorems were those that remained invariant under linear transformations of these numbers. Now even the initial definitions are invariant, and vector spaces are defined by axioms rather than as particular objects. However, it is

proved that every vector space has a base, so that the new theory is much the same as the old. But now no particular base is distinguished, and usually arguments which use particular bases are cumbrous and inelegant compared to arguments directly in terms of the axioms.

We believe that mathematics itself can be founded in an invariant way, which would be equivalent to, but would not involve, formalisation within some theory like ZF. No particular axiomatic theory like ZF would be needed, and indeed attempts to force arbitrary theories into a single formal strait-jacket will probably continue to produce unnecessarily cumbrous and inelegant contortions.

For those who doubt the possibility of such a programme, it might be worthwhile to note that certainly principles (i) and (ii) of our Mathematicians' Lib movement can be expressed directly in terms of the predicate calculus without any mention of sets (for instance), and it can be shown that any theory satisfying the corresponding restrictions can be formalised in ZF together with sufficiently many axioms of infinity.

Finally, we note that we have adopted the modern habit of identifying ZF (which properly has only sets) with the equiconsistent theory NBG (which has proper Classes as well) in this appendix and elsewhere. The classification of objects as Big and small is not peculiar to this theory, but appears in many foundational theories, and also in our formalised versions of principles (i) and (ii).